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FUNDAMENTALS FOR THE GENERATION OF A MATH MODEL FOR A MISSILE FLYING IN THREE-DIMENSIONAL SPACE WITH SIX DEGREES OF FREEDOM

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REAL-TIME FLIGHT ANALYSIS

FUNDAMENTALS FOR THE GENERATION OF A MATH MODEL FOR A MISSILE IN THREE-DIMENSIONAL SPACE WITH SIX DEGREES OF FREEDOM.

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September 1960

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A B S T R A C T

Methods are explained for deriving the differential equations describing a general missile airframe flying in three-dimensional space. All the forces and moments acting on the missile are written as vectors. These vectors are readily expressed in suitable coordinate systems. After they are so expressed, they are transformed to a common coordinate system where the solutions are carried out. The desired output information is then transformed to a fixed system on the ground and presented in terms of that system. A short exposition of vector analysis, to the extent required, is included, and matrix notation is emphasized throughout.

Methods (utilizing vector analysis and coordinate system transformations) are presented for handling thrust misalignment, the general moment of inertia tensor, curvature of the earth, rotation of the earth, targets, etc.

The methods, as employed, generate an exact representation of the flying missile; however, wherever possible, approximations are suggested which simplify the equations. Errors, not only from approximations, but from other sources, are pointed out when such situations arise.

Since there are so many types of guidance philosophies, only very general statements are made concerning guidance. However, the guidance philosophies for two specific types of missiles are included as examples.

C O N T E N T S

FUNDAMENTALS FOR THE GENERATION OF A MATH MODEL FOR A MISSILE IN THREE-DIMENSIONAL SPACE WITH SIX DEGREES OF FREEDOM.

INTRODUCTION

The following series of notes gives the derivation for the general equations which describe any missile system in three dimensional space. Many people have expressed a desire to have these fundamentals expounded in detail and also have the derivations, themselves, in printed form for future reference; hence this series.

These notes, which will also be given as lectures, have previously been given to a select group in a sort of "dry run". That group felt that these notes would provide a valuable reference, and could be used also as a fundamental basis in deriving equations for any missile system, since the equations for a general missile system are of the same type no matter what type of missile is used. The airframe equations (which these lectures are primarily concerned with) are more or less fundamentally identical in every case of a flying missile. We are dealing with a self-powered free-free beam in space which is acted upon by aerodynamic, propulsive, and gravitational forces. No matter what type of missile we are using the same forces will be acting on it. The method for obtaining the accelerations and trajectory follow the basic physics of the free-free beam configuration.

Of course, by three dimensional space we mean the atmosphere which surrounds us. The phrase "six degrees of freedom" describes the number of equations required to solve this system: three for the solution of the forces and three more which give solutions for the moments. From these, we are able to obtain enough independent equations to determine completely the unknown quantities concerning the missile. The term free-free beam mentioned earlier simply describes the appearance of a flying missile.

We will begin with the very basic fundamentals and, step by step, accrue the necessary terms to completely describe the missile system. In many cases the information which is contained in this document may be something which the reader knows well and it will be a redundancy. However, for many others who have not had the opportunity to meet this exposition, this will not be the case.

The basic organization will proceed from the development of the force and moment equations and the development of the coordinate systems necessary to describe the forces and moments, to the matrix transformations which will be employed. Along the way we will introduce the concept of relative wind and methods by which we may study such things as thrust misalignment. The second lecture period will be devoted to describing the aspects of vector analysis which we will need for our problems. If the reader or listener is familiar with this vector algebra he may skip this part with no loss of understanding. It is included here primarily to review and remind us what part of the large body of knowledge called vector analysis we will use.

DEVELOPMENT OF THE MATHEMATICAL MODEL

FORCE EQUATIONS

Whenever we wish to solve equations concerning any missile system we are primarily interested in determining how high or how far the missile will go. The final result we wish to obtain from any series of equations are the ranges and/or velocities at any instant of time. If we can obtain expressions which determine the position of the missile with respect to some fixed place, we will have the solution to almost all problems with which we will be dealing; certainly, those concerned with trajectory information.

We will begin by describing the forces acting on the missile. To do this, we will solve Newton's so-called second law of motion, the force equation. In vector notation this is:

$$\Sigma \vec{F} = m\vec{A}$$

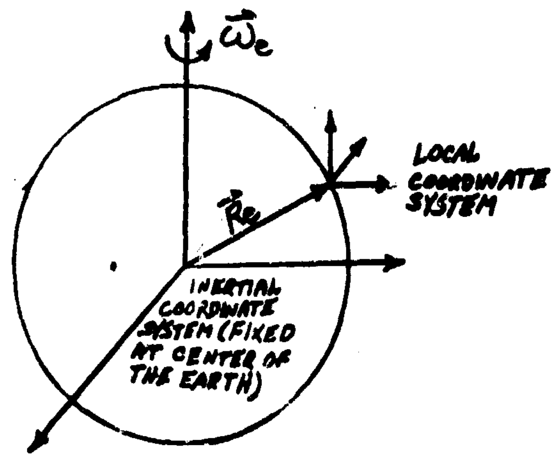
where the vector $\Sigma \vec{F}$ is the total force, m is the mass of the missile at any instant, and vector $\vec{A}$ is the acceleration. We can immediately break this equation down into the following form.

$$\Sigma \vec{F} = \vec{F}_{gr} + \vec{F}_A + \vec{F}_T + \vec{F}_{OTHER}$$

In this expression $\vec{F}_{gr}$ is the gravitational force, the next term is the aerodynamic force, followed by the thrust force. To make things complete, a term is put in called $\vec{F}_{OTHER}$ in case our missile has some odd type of force which is not contained in the gravitational, aerodynamic, or thrust terms. For example, a rotating missile experiences a side force called the Magnus Effect--a force which would be included in "$\vec{F}_{OTHER}$". In general the only forces acting on the missile are the gravitational force, the aerodynamic force, and the power or thrust force. Throughout these lectures we will assume that there is no "other" force. Such forces as Coriolis force and centrifugal force are fictional forces which merely describe the transformation between a non-accelerated coordinate system and one which is accelerated. In this derivation there will be no necessity for ever examining such "fictional" forces specifically. If we wish to actually look at them, they can be found in the equations, but they arise naturally out of the transformations which we will use in deriving the equations.

Newton's second law, as stated above, is only true (disregarding relativistic effects) in an "inertial" system. By an "inertial" system, we mean one which is not accelerated. Therefore, we must designate some place, preferably where we are standing, as being unaccelerated. Generally, this inertial system is put on the ground, and so in this series we are going to call our inertial system the "ground system". Actually, for a complete and accurate description of the flying missile, we should put our inertial system at the center of the earth, because the earth rotates on its axis.

Thus:



In this case, the earth would rotate around the "inertial" system which remains fixed at the center of the earth. The point of launch, then, would rotate with the earth with velocity equal to the tangential velocity of the earth's surface. However, for ease of derivation, we are going to assume that the earth does not rotate. (Since the rotation is slow, in many cases the induced error is small enough to be ignored.) If it becomes necessary (and it would if we were working with long range missiles) we could put our inertial system at the center of the earth and the transformations to be subsequently described would be of exactly the same type. Since this series of lectures is intended to impart information, we will not go into the detail of the transformations from the center of the earth. The only thing that this type of transformation would do would be to give more complexity to the resulting equations but add nothing to the theory. Another thing that we are going to assume is that the earth is flat. Generally, the error, because of this, is not great if the trajectory is relatively short. If the trajectory were long, it would require that we make corrections to our equations to take into account the curvature of the earth. Again, this merely makes the resulting equations more complicated and, since it adds nothing particularly to the theory, we will not go into this aspect of the problem.

We will begin by assuming our inertial coordinate system to be fixed to the earth at the place of launch of the missile. The inertial system, or the ground system as we will call it, is a right-handed orthogonal cartesian set of axes. They will be defined by three unit vectors $\bar{\xi}_1$, $\bar{\xi}_2$, and $\bar{\xi}_3$. Before we can proceed any further we must define what we mean by positive and negative translations and rotations. If we translate in a positive direction, we move from the center or origin of the coordinate system outwards in the direction which a unit vector points. The following diagram will illustrate what we mean by positive rotation:

If we rotate about any of the three unit vectors, or axes, then the rotation will be positive if, when we are rotating about the first unit vector, $\vec{g}_1$, the rotation is from unit vector number two toward unit vector number three. If we rotate about the second unit vector, the rotation is positive if $\vec{g}_3$ rotates toward $\vec{g}_1$. Rotation is positive about the third vector if the rotation is in the sense from $\vec{g}_1$ to $\vec{g}_2$. Therefore, whenever we rotate if we always rotate in these directions, that is, from one to two, two to three, and three to one, then our rotations will be positive, and all the equations will automatically have the correct sign.

We are now in a position to further define the forces in the previous equation. For example the gravitational force always points to the center of the earth. If we assume that our third ground axis, that is, $\vec{g}_3$, is pointing upward from the center of the earth then the gravitational force is given as:

$$\vec{F}_{gr} = -mg\vec{g}_3$$

The aerodynamic forces, on the other hand, are related specifically to the direction which the missile is flying. It would be extremely difficult to describe the aerodynamic forces in terms of the ground system which we have just defined above. Therefore, for this purpose, we will define a new coordinate system called the wind system. It is another orthogonal cartesian coordinate system with three unit vectors $\vec{w}_1$, $\vec{w}_2$, and $\vec{w}_3$. In this system we write down, exactly, the aerodynamic forces in our equations. These are written as:

$$\vec{F}_A = D\vec{w}_1 + S\vec{w}_2 + L\vec{w}_3$$

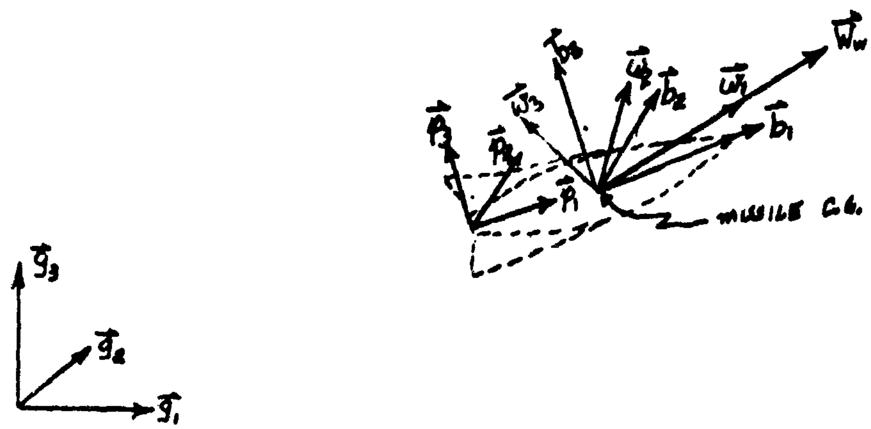
In the wind system the terms used for the aerodynamic force are drag, side force, and lift, and these terms refer only to aerodynamic forces defined in the wind system. This wind system is so set up, that the first unit vector $\vec{w}_1$ always points in the direction of the relative wind vector. The other two vectors $\vec{w}_2$ and $\vec{w}_3$ are such that we can rotate through two angles, called angles of attack, and these vectors will be oriented with the body of the missile.

The thrust force, $\vec{F}_{T}$, will be defined in a system called the thrust system. The coordinates will be designated by three orthogonal unit vectors $\vec{p}_1$, $\vec{p}_2$, and $\vec{p}_3$. We choose "p" rather than "t" for thrust, because "t" will be used as temperature, time, and reference to the target in other equations. Actually we can call "p" the "push" of the missile. We will describe the thrust system, that is, the $\vec{p}_i$'s, so that the force due to the ejection of material from the missile, the thrust, acts in the direction of $\vec{p}_1$. Then we can simply rotate through two angles in order to line up our thrust system with the missile body. If we assume there is no thrust misalignment, then the angles between the "p" system and the missile body will be zero -- and the thrust will be presumed to act through the missile centerline. If we ever wish to investigate thrust misalignment, we will already have the mathematics for doing so.

We will need to describe one other coordinate system whose significance will become more apparent later. This is called the body system, and is designated by the three unit vectors $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$. We are actually going to solve all of our equations in this body system. This system is defined in the following manner: the $\vec{b}_1$, that is the body-1 axis is aligned so that it is along the center line of the missile. We will assume that the origin of this system is

at the center of gravity of the missile. The other two vectors, $\vec{b}_2$ and $\vec{b}_3$, can be defined more or less at the analyst's discretion. However, it is apparent, or at least will be apparent, that certain types of orientations are better than others.

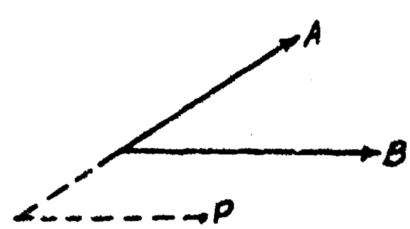
The following diagram illustrates the coordinate systems which have just been described: those coordinate systems which we called the ground system, the body system, the thrust system and the wind system.



VECTORS

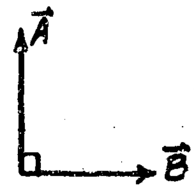
Unit Vectors

In the work which we are concerned with, that is the description of missiles, their trajectories, and responses, the most powerful mathematical tool we can use is vector analysis. In any space, say two-dimensional or three-dimensional in which we are generally concerned in this type of study, we can define the position of any point by means of two lines on a plane or by three lines in a three-dimensional space, so long as those lines do not coincide. For example, if we draw two lines in a plane as in the following diagram,

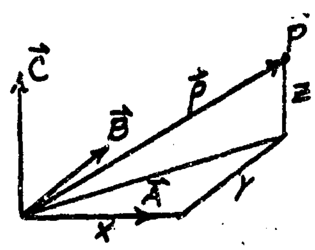


then we can get to any point on this plane by moving only parallel to these two lines. In the above diagram we went negatively along A and then went positively along B, where positive is in the direction of the arrows. If we set up a suitable scale on the lines A and B so that they have a definite length, and if we decide that they point in a certain direction, then these are called vectors. If the lines A and B were of unit length (in any measuring system) they are called unit vectors. Furthermore if we made

these vectors orthogonal, that is at right angles, as in the following diagram:

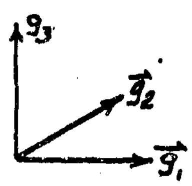


then these two vectors of length one and at right angles to each other are called orthonormal vectors. In three dimensional space we will draw one more vector of unit length which is orthogonal to both $\vec{A}$ and $\vec{B}$ and we will have a three dimensional system with the third vector $\vec{C}$. We can now describe any point in space with respect to these unit vectors. Thus the vector $\vec{P}$ in the following diagram would be written in this system as: $P = x\vec{A} + y\vec{B} + z\vec{C}$,



which simply tells us to travel x units along $\vec{A}$, y units along $\vec{B}$, and z units along $\vec{C}$ to arrive at point P.

Going back to our original ground system, we have three unit vectors, $\vec{g}_1$, $\vec{g}_2$, and $\vec{g}_3$, mutually orthogonal:



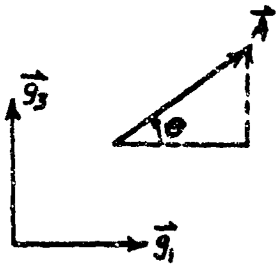
These vectors form an orthonormal coordinate system. We are also going to restrict ourselves to right-handed coordinate systems. A right-handed coordinate system is defined as follows:

> The direction of $\vec{g}_3$ is the direction of advance of a right-handed screw when it is rotated from $\vec{g}_1$ towards $\vec{g}_2$. The direction of $\vec{g}_2$ is the direction of advance of a right-handed screw when rotated from $\vec{g}_3$ towards $\vec{g}_1$. The direction of $\vec{g}_1$ is the direction of advance of a right-handed screw when rotated from $\vec{g}_2$ towards $\vec{g}_3$.

It is possible to use a left-handed coordinate system. Indeed, in Russia the left-handed coordinate system is the norm. However throughout the United States,

England and much of Europe the standard is the right-handed coordinate system, it is the one we will use here, and it is the one in which it is most easy to visualize our functions.

Assume we have defined an orthonormal right-handed coordinate system and we have some vector in space called vector $\vec{A}$.



In the above diagram $\vec{A}$ has some length and it is oriented in a certain way with respect to the coordinate system. To simplify our analysis we will only talk about two unit vectors $\vec{g}_1$ and $\vec{g}_3$. If the length of vector $\vec{A}$ is "A" units and the angle between a line parallel to the $\vec{g}_1$ axis and the vector $\vec{A}$ is θ, then the length of the line parallel to the $\vec{g}_1$ axis is equal to $|\vec{A}| \cos \theta$, and the length of the line parallel to the $\vec{g}_3$ axis is $|\vec{A}| \sin \theta$. Therefore, we can say that the vector $\vec{A}$ is $|\vec{A}| \cos\theta \, \vec{g}_1 + |\vec{A}| \sin\theta \, \vec{g}_3$ where the absolute value, $|\vec{A}|$ represents the magnitude of the vector without any indication of direction. The direction of vector $\vec{A}$ is given by $\vec{g}_1$ and $\vec{g}_3$.

Addition and Subtraction of Vectors

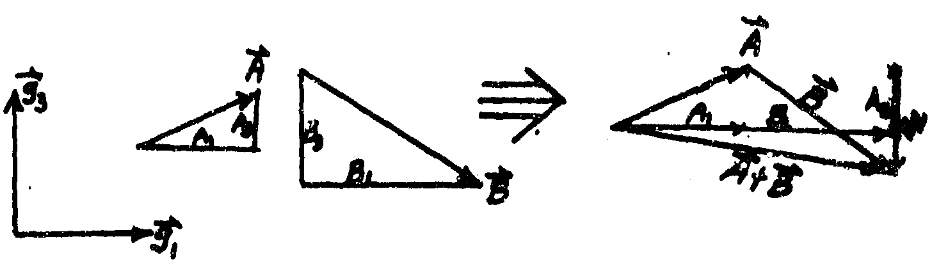
Assume we have two vectors $\vec{A}$ and $\vec{B}$ which have components in the $\vec{g}_1$ system as:

$$\vec{A} = A_1 \vec{g}_1 + A_2 \vec{g}_2 + A_3 \vec{g}_3$$
$$\vec{B} = B_1 \vec{g}_1 + B_2 \vec{g}_2 + B_3 \vec{g}_3$$

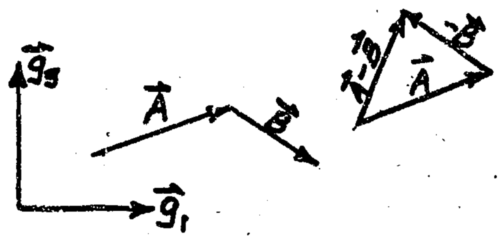
These vectors can be manipulated by means of vector algebra. For instance, two vectors can be added. The addition of $\vec{A}$ and $\vec{B}$ is defined as follows:

$$\vec{A} + \vec{B} = (A_1 + B_1)\vec{g}_1 + (A_2 + B_2)\vec{g}_2 + (A_3 + B_3)\vec{g}_3$$

The result of adding the two vectors $\vec{A}$ and $\vec{B}$ is that we add the components of the $\vec{g}_1$ vector together (the distance we go in the direction of $\vec{g}_1$.) Similarly, we add the components of $\vec{g}_2$ and $\vec{g}_3$. We can demonstrate the addition of these two vectors as in the following diagram (assuming $\vec{A}$ and $\vec{B}$ only have $\vec{g}_1$ and $\vec{g}_3$ components):



It can be easily seen that adding the components of the vectors $\vec{A}$ and $\vec{B}$, to each other is tantamount to adding the two vectors with their respective directions tail to head and drawing a line between the tail of the first to the head of the second. The above diagram illustrates another important principle: the actual position in space of these vectors is immaterial. We can move them anywhere we want as long as we keep their directions and magnitudes the same. Thus we could move the $\vec{B}$ vector up and attach it to the $\vec{A}$ vector to produce the resultant vector. The resultant of adding these two vectors is a third vector, $\vec{A}+\vec{B}$. Subtraction of vectors would be accomplished simply by changing the sign of the components. As can be seen by a simple sketch, this is the same as if we reversed the direction of the subtracted vector and did not interfere with the magnitude:



The Dot Product:

Multiplication of vectors can be performed in two different ways. One of these is the dot product or scalar product, the other is the cross or vector product. The dot product gives a scalar number, that is, a number having a magnitude but no direction. The cross product gives a new vector. In each case we perform a type of multiplication between two vectors or perhaps among several vectors. The dot product can be defined in two ways. Using the vectors $\vec{A}$ and $\vec{B}$ described above we can define the dot product between them as:

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$

where θ is the angle between the vectors $\vec{A}$ and $\vec{B}$. If we write the vectors $\vec{A}$ and $\vec{B}$ in their component form, then the dot product between the two is the regular scalar or number product of each component and then the products of these components are summed together as below:

$$\vec{A} = A_1\vec{g}_1 + A_3\vec{g}_3$$

$$\vec{B} = B_1\vec{g}_1 + B_3\vec{g}_3$$

$$\vec{A}\cdot\vec{B} = A_1B_1 + A_3B_3$$

In the type of analysis which we plan on doing, the most important type of dot product is that between unit vectors. Going back to the first definition

let us take a dot product as we would a regular algebraic product between two polynomials. Thus:

$$(A_1\vec{g}_1+A_3\vec{g}_3)\cdot(B_1\vec{g}_1+B_3\vec{g}_3)=A_1\vec{g}_1\cdot B_1\vec{g}_1+A_1\vec{g}_1\cdot B_3\vec{g}_3+A_3\vec{g}_3\cdot B_1\vec{g}_1+A_3\vec{g}_3\cdot B_3\vec{g}_3$$

At this point we should mention that the scalars multiplying vectors can be factored and moved to the front of the expression. Thus the first term in the above equation can be written as $A_1B_1\vec{g}_1\cdot\vec{g}_1$. We know how to multiply real numbers, so we must find the dot product between $\vec{g}_1$ and $\vec{g}_1$. In each of the above terms if we use the first definition of the dot product we see that we have the following form for the product of these two vectors:

$$\vec{A}\cdot\vec{B}=A_1B_1|\vec{g}_1||\vec{g}_1|\cos\theta+A_1B_3|\vec{g}_1||\vec{g}_3|\cos\phi+A_3B_1|\vec{g}_3||\vec{g}_1|\cos\phi+A_3B_3|\vec{g}_3||\vec{g}_3|\cos\theta$$

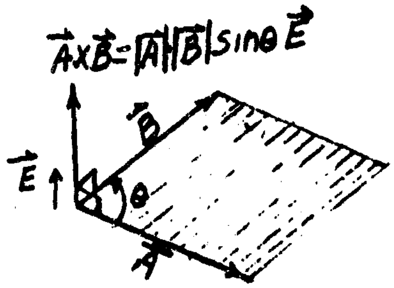
We see that the dot product between $\vec{g}_1$ and $\vec{g}_1$ is the product of the magnitudes of $\vec{g}_1$ times the cosine of the angle between them. In this case the magnitudes are one times one and the angle between $\vec{g}_1$ and $\vec{g}_1$ is, of course, zero. Therefore, $\vec{g}_1\cdot\vec{g}_1$ is equal to one. On the other hand the product $\vec{g}_1\cdot\vec{g}_3$ is one times one, times the cosine of the angle between $\vec{g}_1$ and $\vec{g}_3$. We have defined our coordinate system so the $\vec{g}_1$ is orthogonal to $\vec{g}_3$ and therefore, the angle between $\vec{g}_1$ and $\vec{g}_3$ is 90°. The cosine of 90° is zero. Thus, $\vec{g}_1\cdot\vec{g}_3$ and $\vec{g}_3\cdot\vec{g}_1$ equal zero. The only terms left are the first and last terms and the value of the dot product is $A_1B_1+A_3B_3$. This allows us to state in words a rule for the dot products of vectors when defined in orthonormal systems: "The dot product between unit vectors equals one if the unit vectors are the same and it equals zero if they are not the same."

The Cross Product

The definition of the cross product is:

$$\vec{A}\times\vec{B}=|\vec{A}||\vec{B}|\sin\theta\,\vec{E}\ ^*$$

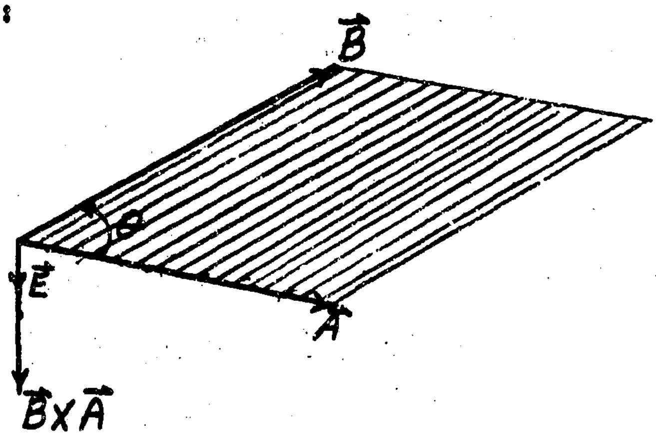
That is, the cross product of $\vec{A}$ and $\vec{B}$ is the scalar product of the magnitude of $\vec{A}$ times the magnitude of $\vec{B}$ times the sine of the angle between them. The direction $\vec{E}$ is a unit vector orthogonal to the plane of $\vec{A}$ and $\vec{B}$. Thus:



Furthermore, the direction of $\vec{E}$ is pointing in the direction of the advance of a right-handed screw if we rotate $\vec{A}$ towards $\vec{B}$.

* The fact that we have a vectorial result is not an accident, but the proof relies on tensor analysis and will not be dealt with here.

The product of $\vec{B}$ cross $\vec{A}$, then, would be in the direction of the advance of a right-handed screw if we rotate $\vec{B}$ towards $\vec{A}$. This can be seen from the following diagram:



Thus we can say that:

$$\vec{A}\times\vec{B}=-\vec{B}\times\vec{A}$$

The cross product between unit vectors, which we will generally be concerned with, is as follows:

From the definition, $\vec{A}\times\vec{B}=|\vec{A}|\cdot|\vec{B}|\sin\theta\,\vec{E}$;, if $\vec{A}=\vec{B}$ then $\theta=0$ and $\vec{A}\times\vec{B}=\vec{0}$. The cross products of the $\vec{g}_i$ vectors are:

$$\vec{g}_1\times\vec{g}_1=\vec{g}_2\times\vec{g}_2=\vec{g}_3\times\vec{g}_3=0$$

$$\vec{g}_1\times\vec{g}_2=\vec{g}_3$$

$$\vec{g}_2\times\vec{g}_3=\vec{g}_1$$

$$\vec{g}_3\times\vec{g}_1=\vec{g}_2$$

and, alternatively,

$$\vec{g}_2\times\vec{g}_1=-\vec{g}_3$$

$$\vec{g}_3\times\vec{g}_2=-\vec{g}_1$$

$$\vec{g}_1\times\vec{g}_3=-\vec{g}_2$$

As we stated above, the cross product produces a new vector pointing in the direction of the advance of a right-handed screw, if we rotate the first vector, in the term, toward the second. From the definition of positive rotation and orthonormality, the above definitions for the $\vec{g}_i$'s are immediate.

A convenient method of determining the cross product of two vectors is by using the determinant defined below:

Given $\vec{A}=A_1\vec{g}_1+A_2\vec{g}_2+A_3\vec{g}_3$, $\vec{B}=B_1\vec{g}_1+B_2\vec{g}_2+B_3\vec{g}_3$

$$\text{Then } \vec{A}\times\vec{B}=\begin{vmatrix}\vec{g}_1 & \vec{g}_2 & \vec{g}_3\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix}=(A_2B_3-A_3B_2)\vec{g}_1+(A_3B_1-A_1B_3)\vec{g}_2+(A_1B_2-A_2B_1)\vec{g}_3$$

The two definitions for the cross product, that is, the determinant method and the definition, $\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin \theta \, \vec{E}$, are completely equivalent. The proof of this is easily shown by expansion and will not be given here.

Throughout our work in the analysis of missile systems we will be using the dot product between unit vectors and the cross product between unit vectors. This is because we will always describe our forces, our moments and other aspects of the response of the missile in terms of magnitude, with the directions indicated by the unit vectors.

Triple Cross Product and Triple Scalar Product

From time to time vector products of the form $\vec{A}\times(\vec{B}\times\vec{C})$ arise. Such a vector is called the triple cross product or triple vector product. That the triple cross product is not necessarily associative is very important. That is, $(\vec{A}\times\vec{B})\times\vec{C} \neq \vec{A}\times(\vec{B}\times\vec{C})$. This is very easy to show. Suppose that

$$\vec{B} = B_1\vec{g}_1 + B_2\vec{g}_2 + B_3\vec{g}_3$$
$$\vec{C} = C_1\vec{g}_1 + C_2\vec{g}_2 + C_3\vec{g}_3$$

Then to solve the product $\vec{B}\times(\vec{B}\times\vec{C})$, we first find:

$$\vec{B}\times\vec{C} = \begin{vmatrix} \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = (B_2C_3 - B_3C_2)\vec{g}_1 + (B_3C_1 - B_1C_3)\vec{g}_2 + (B_1C_2 - B_2C_1)\vec{g}_3$$

which gives:

$$\vec{B}\times(\vec{B}\times\vec{C}) = \begin{vmatrix} \vec{g}_1 & \vec{g}_2 & \vec{g}_3 \\ B_1 & B_2 & B_3 \\ B_2C_3 - B_3C_2 & B_3C_1 - B_1C_3 & B_1C_2 - B_2C_1 \end{vmatrix}$$

Solving:

$$\vec{B}\times(\vec{B}\times\vec{C}) = \left[B_2(B_1C_2 - B_2C_1) - B_3(B_3C_1 - B_1C_3)\right]\vec{g}_1 + \left[B_3(B_2C_3 - B_3C_2) - B_1(B_1C_2 - B_2C_1)\right]\vec{g}_2$$
$$+ \left[B_1(B_3C_1 - B_1C_3) - B_2(B_2C_3 - B_3C_2)\right]\vec{g}_3$$

$$= (B_1B_2C_2 - B_2^2C_1 - B_3^2C_1 + B_1B_3C_3)\vec{g}_1 + (B_2B_3C_3 - B_3^2C_2 - B_1^2C_2 + B_1B_2C_1)\vec{g}_2$$
$$+ (B_1B_3C_1 - B_3^2C_2 - B_1^2C_2 + B_2B_3C_2)\vec{g}_3$$

But from $(\vec{B}\times\vec{B})\times\vec{C}$ we know that $\vec{B}\times\vec{B} = 0$ for any values of $\vec{B}$ or $\vec{C}$. Therefore, except for the case $\vec{B}\times(\vec{B}\times\vec{C}) = 0$, the vector triple product is not associative.

The triple scalar product is of the type:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) \quad \text{or} \quad (\vec{A}\times\vec{B})\cdot\vec{C}$$

We can immediately remove the parentheses since $(\vec{A}\cdot\vec{B})\times\vec{C}$ or $\vec{A}\times(\vec{B}\cdot\vec{C})$ is meaningless. Also $\vec{A}\cdot\vec{B}\times\vec{C} = \vec{B}\times\vec{C}\cdot\vec{A}$ since the dot product is commutative.

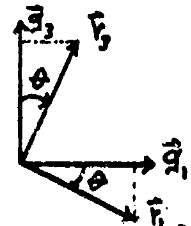
Next we can show $\vec{A}\cdot\vec{B}\times\vec{C} = \vec{C}\cdot\vec{A}\times\vec{B} = \vec{B}\cdot\vec{C}\times\vec{A}$, for:

$$\vec{A}\cdot\vec{B}\times\vec{C} = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \vec{A}\times\vec{B}\cdot\vec{C} = \vec{B}\cdot\vec{C}\times\vec{A}$$

The cross product is distributive. That is, $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$. This case can be easily demonstrated by expanding the two sides of the equation.

DERIVATIVES IN ROTATING COORDINATE SYSTEMS

The following diagram illustrates two coordinate systems, a $\vec{g}_1$ system and an $\vec{r}_1$ system, in which the $\vec{r}_1$ system has been rotated with respect to $\vec{g}_1$ system:



In each of these systems the vectors $\vec{g}_1$, $\vec{g}_3$, $\vec{r}_1$, and $\vec{r}_3$ have unit length. However, if we represent the length of the vector $\vec{r}_1$ in the $\vec{g}_1$ system, then, as is indicated in the diagram, the component of $\vec{r}_1$ parallel to $\vec{g}_1$ has a magnitude less than 1, and similarly for the component of $\vec{r}_1$ parallel to $\vec{g}_3$. Thus, if we are dealing with two coordinate systems, one of which has been rotated with respect to the other, the magnitudes of the unit vectors in one system will not be unity if described in the other. Let us take the derivative of a vector described in the $\vec{r}_1$ system, vector $\vec{A}$. Where $\vec{A}$ is:

$$\vec{A} = A_1\vec{r}_1 + A_3\vec{r}_3$$

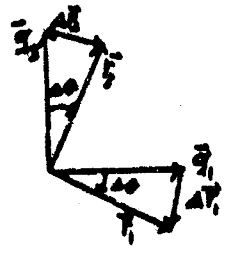
and

$$\frac{d\vec{A}}{dt} = \frac{dA_1}{dt}\vec{r}_1 + A_1\frac{d\vec{r}_1}{dt} + \frac{dA_3}{dt}\vec{r}_3 + A_3\frac{d\vec{r}_3}{dt}$$

or:

$$\frac{dA}{dt} = (\dot{A}_1\vec{r}_1 + \dot{A}_3\vec{r}_3) + (A_1\dot{\vec{r}}_1 + A_3\dot{\vec{r}}_3)$$

From this grouping we see that the derivative of $\vec{A}$ is the derivative of the components along each axis still in the direction of that axis, plus the derivative of the axes themselves. We now wish to obtain an expression for determining the derivative of the unit vectors. From the following diagram:



If $\vec{r}_1$ coincided with $\vec{g}_1$ and $\vec{r}_3$ coincided with $\vec{g}_3$, then after a rotation through $\Delta\theta$, the new $\vec{r}_1$ and $\vec{r}_3$, say $\vec{r}_1'$ and $\vec{r}_3'$, would be:

$$\vec{r}_1' = \vec{r}_1 + \Delta\vec{r}_1 \quad \text{and} \quad \vec{r}_3' = \vec{r}_3 + \Delta\vec{r}_3$$

If $\Delta\theta$ is very small, then we can write the approximations:

$$\Delta\vec{r}_1 = -|\vec{r}_1| \sin\Delta\theta\, \vec{r}_3 \quad \text{and} \quad \Delta\vec{r}_3 = |\vec{r}_3| \sin\Delta\theta\, \vec{r}_1$$

or:

$$\Delta\vec{r}_1 = -\Delta\theta\, \vec{r}_3 \quad \text{and} \quad \Delta\vec{r}_3 = \Delta\theta\, \vec{r}_1$$

Dividing by Δt gives:

$$\frac{\Delta\vec{r}_1}{\Delta t} = -\vec{r}_3 \frac{\Delta\theta}{\Delta t} \quad \text{and} \quad \frac{\Delta\vec{r}_3}{\Delta t} = \vec{r}_1 \frac{\Delta\theta}{\Delta t}$$

Which in the limit as $\Delta t \to 0$ is exactly:

$$\frac{d\vec{r}_1}{dt} = -\vec{r}_3 \frac{d\theta}{dt} \quad \text{and} \quad \frac{d\vec{r}_3}{dt} = \vec{r}_1 \frac{d\theta}{dt}$$

This gives us an expression for the derivatives of the unit vectors. However, there is a more convenient method for determining this. If ω is the angular speed of rotation then:

$$\Delta\theta = \omega \Delta t$$

Define the vector angular velocity as:

$$\vec{\omega} = \omega \vec{r}_2,$$

that is, its direction is along the axis of rotation. The cross product of $\vec{\omega}$ and the various $\vec{r}_i$ vectors is:

$$\vec{\omega} \times \vec{r}_1 = \begin{vmatrix} r_1 & r_2 & r_3 \\ 0 & \omega & 0 \\ 1 & 0 & 0 \end{vmatrix} = -\omega\vec{r}_3$$

and

$$\vec{\omega} \times \vec{r}_3 = \begin{vmatrix} r_1 & r_2 & r_3 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{vmatrix} = \omega\vec{r}_1$$

But by definition:

$$\frac{d\vec{r}_1}{dt} = -\omega\vec{r}_3 \quad \text{and} \quad \frac{d\vec{r}_3}{dt} = \omega\vec{r}_1$$

Therefore, we can write:

$$\frac{d\vec{r}_1}{dt} = \vec{\omega}\times\vec{r}_1, \quad \frac{d\vec{r}_3}{dt} = \vec{\omega}\times\vec{r}_3,$$

and by extension:

$$\frac{d\vec{r}_2}{dt} = \vec{\omega}\times\vec{r}_2$$

Returning to the equation for the derivative of $\vec{A}$:

$$\frac{d\vec{A}}{dt} = (\dot{A}_1\vec{r}_1 + \dot{A}_3\vec{r}_3) + (A_1\dot{\vec{r}}_1 + A_3\dot{\vec{r}}_3)$$
$$= (\dot{A}_1\vec{r}_1 + \dot{A}_3\vec{r}_3) + (-\omega A_1\vec{r}_3 + \omega A_3\vec{r}_1)$$
$$= (\dot{A}_1 + \omega A_3)\vec{r}_1 + (\dot{A}_3 - \omega A_1)\vec{r}_3$$

Before we end this part of the discussion, let us look at the equation for $\omega \times \vec{A}$:

$$\vec{\omega}\times\vec{A} = \begin{vmatrix} \vec{r}_1 & \vec{r}_2 & \vec{r}_3 \\ 0 & \omega & 0 \\ A_1 & 0 & A_3 \end{vmatrix} = \omega A_3\vec{r}_1 - \omega A_1\vec{r}_3$$

But this is exactly the part of the derivative due to the rotation of the coordinate system. Thus:

$$\frac{d\vec{A}}{dt} = \dot{A}_1\vec{r}_1 + \dot{A}_2\vec{r}_2 + \dot{A}_3\vec{r}_3 + \vec{\omega}\times\vec{A}$$

We are now in a position to make rules concerning the derivative of a vector. First we have the derivative of the scalar components of the vector--in terms of the rotating coordinate system. To this we must add the cross product of the angular velocity of the rotating coordinate system times the vector itself, as defined in the rotating system. This gives us an equation of the form:

$$\frac{{}^{g}d\vec{A}}{dt} = \frac{{}^{r}d\vec{A}^{r}}{dt} + \vec{\omega}^{r} \times \vec{A}^{r}$$

where the superscripts "g" and "r" mean that the derivatives are to be taken in the g_i and r_i systems, respectively; and the superscripts on the right mean that the components of the vector are to be expressed in the indicated system.

We observed, in the foregoing analysis, that something which rotates can be described effectively by assigning its vector direction along the axis of rotation. If the rotation is positive, it would advance a right-handed screw. This was previously defined as positive in the coordinate system. Of course we can immediately define a negative rotation as we did above.

The above analysis covers all of the essential points which are likely to be encountered with the use of vector analysis in the study of missile systems. The algebra and the study of vectors covers a far wider field than we have done above.* However, we rarely need to get into the other aspects. It is likely that we will encounter situations where we need a greater knowledge of vector analysis, but most of the analysis for the type of work with which we are concerned here, can be done by observing the few simple rules as we have defined them.

EULER ANGLE TRANSFORMATION

So far in our analysis of missile systems we have only described an equation for the forces. In this equation we needed to determine three types of forces. The gravitational force could be immediately written as: $-mg\,\vec{k}_3$. We presume that we know the mass of the missile; certainly g, the constant acceleration due to gravity, is known, so the gravitational term is completely defined. However, we do not know the thrust as such, now. Whether or not we choose to put the thrust in a thrust system with finite angles of misalignment, we still must determine the thrust in the body system. In general, we are given some sort of curve or equation which will describe the magnitude of this thrust vector. If we put the thrust in the body system (thereby assuming that the angles between the thrust coordinate system and the body system are zero) we still

*For those who wish to pursue the subject further, there are many excellent treatments. For example see Vector and Tensor Analysis by Harry Lass, McGraw Hill Book Co., 1950.

must achieve some method of describing the effect of the thrust on the body and the consequent maneuvers that the missile will undergo.

The missile body will be flying an arched trajectory; it will be moving from side to side and it may even be rolling back and forth, or, in some cases, spinning around its own axis. Thus we need to have some sort of mathematical method for determining the orientation of the missile body at any instant of time with respect to our fixed unchanging axes on the ground. We will develop the attitude angles between the missile and the ground by means of what is known as an Euler angle sequence. Mathematically, we assume that we know the amount of angular rotation the body has undergone at any time. Then, if we are able to completely describe all subsequent forces which cause the missile to rotate (thereby moving its fixed body coordinates with it) we are able to keep track of the change in these angles, and we will always know the orientation of the body with respect to the ground.

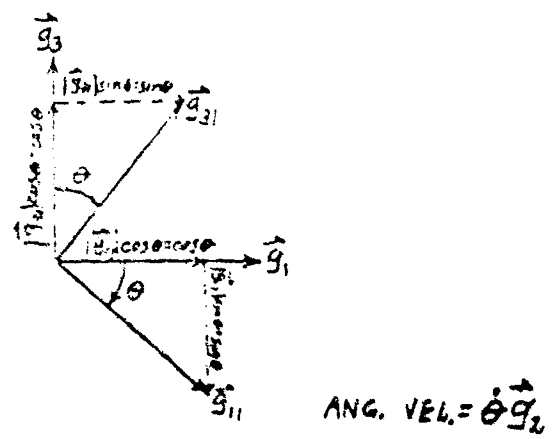
Standard usage throughout this country in the various airplane and missile industries has dictated that certain symbols are to be used for certain angles. Readers who have used spherical coordinates or cylindrical coordinates should realize at this point that the coordinate transformations we are describing are not the same. Although these coordinate transformations essentially convert a rectangular coordinate system through rotary angles to another system it is not the same as transforming from a rectangular system to say, a spherical one. We obtain the body system by a series of rotations; a spherical coordinate system is obtained by projecting various vectors onto planes, the components so projected determining the length, direction, and size of the new coordinate system.

Because of standard usage we will make the following definitions: if we rotate about an axis that is numbered one, as for example the $\vec{g}_1$ axis, or, say, the $\vec{b}_1$ axis, we will call this a "roll" angle. Whenever we rotate around a number two axis, of any system, we will call this a "pitch" angle. In a like manner, any rotation about an axis numbered three, we will call a "yaw" rotation. In the Euler sequence from the ground system to the body system the roll angle is defined as ϕ, the pitch angle is defined as θ, and the yaw angle is defined as ψ. Geometrically, what we are trying to do when we transform coordinates is the following: we start at, say, the ground and rotate our axis through three angles so that they are all pointing in the same directions as the axes of the system to which we are transforming. (In this case the transformed $\vec{g}_1$ unit vector will point in the same direction as the $\vec{b}_1$ unit vector, the transformed $\vec{g}_2$ unit vector in the same direction as the $\vec{b}_2$ vector, and the transformed $\vec{g}_3$ vector will be parallel to the $\vec{b}_3$ vector.) This, then, is actually what we mean by transformation, geometrically.

In physics, as generally taught in school systems, transformation between coordinate systems is performed by a roll, pitch, roll sequence. The reason for this particular type of transformation, I do not know. In our work we will not restrict ourselves to any particular type of transformation. One sequence of angles is as good as another sequence with one restriction. For example, suppose we decide to go through a roll, pitch, yaw sequence. Then the restriction on this sequence is, that the second angle (in this case, θ), can never be near 90°. The reason for this is that the second angle appears

in the final transformation as a tangent. If the second angle nears 90°, then the tangent of this angle approaches infinity. Other than the restriction on the second angle we can take such sequences as θ, ψ, ϕ which is a pitch, yaw, roll sequence. We can take a sequence such as θ, ϕ, ψ. This is a pitch, roll, yaw sequence, etc. No matter which type of sequence we care to use, we must adhere to the definitions of pitch, yaw and roll, that is, the pitch is around a number two axis, a roll is around a number one axis, and yaw is around a number three axis. The methods used for all transformations between the ground and body by an Euler sequence, are the same. Certainly, the method of solving the equations will be similar.

For the purposes of demonstration of the methods employed to transform the ground to body we will use a θ, ψ, ϕ, sequence; that is, a pitch, yaw, roll sequence. This means that we first must rotate the $\vec{g}_1$ system around $\vec{g}_2$. From this we will define the vectors of the subsequent coordinate system in terms of the $\vec{g}_1$ vectors. We will write down the vectors in the following manner: $\vec{g}_{11}$ means the vector $\vec{g}_1$ after the first rotation, $\vec{g}_{21}$ will be the $\vec{g}_2$ vector after the first rotation, and $\vec{g}_{31}$ will be the $\vec{g}_3$ vector after the first rotation. Similarly $\vec{g}_{12}$ will be the $\vec{g}_1$ vector after the second rotation, etc. Also during these transformations, we must keep track of the angular velocities so that we can determine the angular velocities of the missile in terms of the missile coordinate system — that is, the body system. The following diagram demonstrates how we obtain the coordinate system after the first rotation, which in this case, is a pitch rotation.



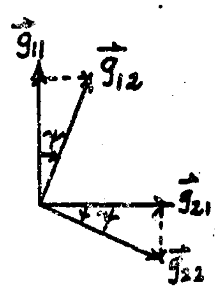
We can write the g_1's in terms of the g_{11}'s as:

$$\vec{g}_{11} = \vec{g}_1 \cos\theta - \vec{g}_3 \sin\theta$$

$$\vec{g}_{21} = \vec{g}_2$$

$$\vec{g}_{31} = \vec{g}_1 \sin\theta + \vec{g}_3 \cos\theta$$

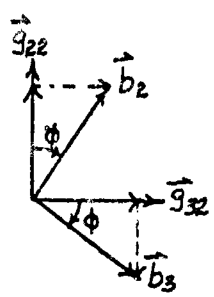
Having obtained the coordinate system which we will call the $\vec{g}_1$ system after the first rotation we will now rotate through the second Euler angle, ψ, a yaw rotation, in the $\vec{g}_{11}$ system. We must rotate about a number three axis, so we will rotate about $\vec{g}_{31}$. The following diagram illustrates the derivation of the coordinate system after the second rotation:



$$\vec{g}_{12} = \vec{g}_{11}\cos\psi + \vec{g}_{21}\sin\psi$$
$$\vec{g}_{22} = -\vec{g}_{11}\sin\psi + \vec{g}_{21}\cos\psi$$
$$\vec{g}_{32} = \vec{g}_{31}$$

$$\text{ANG. VEL.} = \dot{\psi}\,\vec{g}_{31}$$

In order to transform the ground system axes so that the corresponding numbered unit vectors are parallel to the corresponding numbered unit vectors of the body axes, it is necessary to use three distinct angular rotations. We have used two angular rotations so that one more rotation is required to align our ground system with the body system. The final rotation is a ϕ or roll angle. This means we must rotate about $\vec{g}_{12}$ that is, the first $\vec{g}_1$ vector after the second rotation. Since we will now be aligned in the $\vec{b}_1$ system our third rotation is:



$$\vec{b}_1 = \vec{g}_{12}$$
$$\vec{b}_2 = \vec{g}_{22}\cos\phi + \vec{g}_{32}\sin\phi$$
$$\vec{b}_3 = -\vec{g}_{22}\sin\phi + \vec{g}_{32}\cos\phi$$

$$\text{ANG. VEL.} = \dot{\phi}\,\vec{b}_1$$

We now have expressions relating, by a series of rotations, the ground system to the body system. Also we have kept track of the angular velocities for each rotation which we performed.

The form of the Euler angle relationships is awkward to use, so as a compact tool we will introduce the transformation matrix relating the ground and body vectors. In matrix notation, the $\vec{g}_i$ vectors are related to the $\vec{g}_{i1}$ vectors by:

$$\begin{pmatrix} \vec{g}_{11} \\ \vec{g}_{21} \\ \vec{g}_{31} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{g}_1 \\ \vec{g}_2 \\ \vec{g}_3 \end{pmatrix}$$

In a like manner we can represent the vectors of the $\vec{g}_i$'s after the second rotation as:

$$\begin{pmatrix} \vec{g}_{12} \\ \vec{g}_{22} \\ \vec{g}_{32} \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{g}_{11} \\ \vec{g}_{21} \\ \vec{g}_{31} \end{pmatrix}$$

and the $\vec{b}_i$ vectors in terms of the $\vec{g}_i$ vectors after the second rotation as:

$$\begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \vec{g}_{12} \\ \vec{g}_{22} \\ \vec{g}_{32} \end{pmatrix}$$

if we represent these matrix equations in a more compact notation we can write the following equations:

$$(\vec{g}_{i1}) = M(\Theta)(\vec{g}_i)$$
$$(\vec{g}_{i2}) = M(\Psi)(\vec{g}_{i1})$$
$$(b_i) = M(\Phi)(\vec{g}_{i2})$$

Then substituting in the equation for $(\vec{g}_{12})$ and $(\vec{g}_{11})$ we obtain the following equa-

tion:

$$(\vec{b}_i) = M(\phi)M(\psi)M(\theta)(\vec{g}_i)^*$$

Rather than indicate the multiplication of these three matrices together we will give the final result. That is:

$$\begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix} = \begin{pmatrix} \cos\psi\cos\theta & \sin\psi & -\cos\psi\sin\theta \\ -\cos\phi\sin\psi\cos\theta+\sin\phi\sin\theta & \cos\phi\cos\psi & \cos\phi\sin\psi\sin\theta+\sin\phi\cos\theta \\ \sin\phi\sin\psi\cos\theta+\cos\phi\sin\theta & -\sin\phi\cos\psi & -\sin\phi\sin\psi\sin\theta+\cos\phi\cos\theta \end{pmatrix} \begin{pmatrix} \vec{g}_1 \\ \vec{g}_2 \\ \vec{g}_3 \end{pmatrix}$$

Derivation of the Euler Angles

The above matrix identity gives us immediately, presuming we know the size of the the three angles, the relationship between the ground and the body coordinate systems. As everyone is probably already aware, every element of the matrix on the left side of the equal sign must equal the element corresponding in position in the matrix on the right side. For example $\vec{b}_1$, from the product of the two matrices on the right side, is:

$$\vec{b}_1 = \cos\psi\cos\theta\vec{g}_1 + \sin\psi\vec{g}_2 - \cos\psi\sin\theta\vec{g}_3$$

Similarly, we can find expressions for $\vec{b}_2$ and $\vec{b}_3$. Thus if we can write a vector in terms of the body system, we can then employ this transformation matrix and express it in terms of our fixed ground frame. However, we have introduced three more unknown quantities; ψ, θ, ϕ. Therefore, further analysis will require that we find methods and relationships in order to solve for these angles.

The following angular velocity notation for the body system is a standard adopted by most of industry. The angular rate around b_1, the roll rate, is designated by P, the angular rate about $\vec{b}_2$, the pitch rate, is represented by Q, and the yaw rate by R. The angular velocity vector can be considered as a vector in space. We have two coordinate systems, well-defined, with the transformation between them. We can write the angular velocity in terms of either coordinate system, but, because of ease of computation, which we will see later, we will solve for the angular velocity with components in the body system. Since we kept track of the angular velocities during our transformation, we can write:

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*Note: that the transformation matrices as defined can be utilized for other Euler sequences. For example, a ψ, ϕ, θ sequence would be written as $(\vec{b}_1) = M(\theta) M(\phi) M(\psi) (\vec{g}_1)$. The angular velocities would be written, however as: $\dot{\psi}g_3$, $\dot{\phi}g_{11}$, and $\dot{\theta}b_2$.

$$P\vec{b}_1 + Q\vec{b}_2 + R\vec{b}_3 = \dot{\phi}\vec{b}_1 + \dot{\theta}\vec{g}_2 + \dot{\psi}\vec{g}_3 = \dot{\phi}\vec{b}_1 + \dot{\theta}\vec{g}_2 + \dot{\psi}\sin\theta\,\vec{g}_1 + \dot{\psi}\cos\theta\,\vec{g}_3$$

We are now able to solve for the angular rates, as represented in the body system, in terms of trigonometric functions and rates of the Euler angles, themselves. Taking the dot or scalar product on each side of the equation with $\vec{b}_1$ gives:

$$P = \dot{\phi} + \dot{\theta}\,\vec{g}_2\cdot\vec{b}_1 + \dot{\psi}\sin\theta\,\vec{g}_1\cdot\vec{b}_1 + \dot{\psi}\cos\theta\,\vec{g}_3\cdot\vec{b}_1$$

In addition we can find Q by taking the scalar product on each side of the equal sign with $\vec{b}_2$, and we can find R by taking the scalar product with $\vec{b}_3$ on each side of the equal sign. This gives us the following series of three equations:

$$P = \dot{\phi} + \dot{\theta}\,\vec{g}_2\cdot\vec{b}_1 + \dot{\psi}\sin\theta\,\vec{g}_1\cdot\vec{b}_1 + \dot{\psi}\cos\theta\,\vec{g}_3\cdot\vec{b}_1$$

$$Q = \dot{\theta}\,\vec{g}_2\cdot\vec{b}_2 + \dot{\psi}\sin\theta\,\vec{g}_1\cdot\vec{b}_2 + \dot{\psi}\cos\theta\,\vec{g}_3\cdot\vec{b}_2$$

$$R = \dot{\theta}\,\vec{g}_2\cdot\vec{b}_3 + \dot{\psi}\sin\theta\,\vec{g}_1\cdot\vec{b}_3 + \dot{\psi}\cos\theta\,\vec{g}_3\cdot\vec{b}_3$$

From the transformation matrix which we derived we can immediately find the dot products of the vectors necessary to solve these equations. Substituting in the appropriate trigonometric functions for all of the dot products gives us the following expressions for P, Q and R:

$$P = \dot{\phi} + \dot{\theta}\sin\psi + \dot{\psi}\cancel{\sin\theta\cos\psi\cos\theta} - \dot{\psi}\cancel{\cos\theta\cos\psi\sin\theta}$$

$$P = \dot{\phi} + \dot{\theta}\sin\psi$$

$$Q = \dot{\theta}\cos\phi\cos\psi - \dot{\psi}\cancel{\sin\theta\cos\phi\sin\psi\cos\theta} + \dot{\psi}\sin^2\theta\sin\phi + \dot{\psi}\cancel{\cos\theta\cos\phi\sin\psi\sin\theta} + \dot{\psi}\sin\phi\cos^2\theta$$

$$Q = \dot{\theta}\cos\phi\cos\psi + \dot{\psi}\sin\phi$$

$$R = -\dot{\theta}\sin\phi\cos\psi + \dot{\psi}\cancel{\sin\theta\sin\psi\sin\phi\cos\theta} + \dot{\psi}\cos\phi\sin^2\theta + \dot{\psi}\cos^2\theta\cos\phi - \dot{\psi}\cancel{\cos\theta\sin\phi\sin\psi\sin\theta}$$

$$R = -\dot{\theta}\sin\phi\cos\psi + \dot{\psi}\cos\phi$$

Early in these series we mentioned that we would require six degrees of freedom. So far we have investigated three of them. These are the three components of the force vector. The other three degrees of freedom arise from the three components of the momentum equations. The momentum equations give us an expression for obtaining P, Q, and R. Therefore if we assume we can find P, Q, and R we can use the above equations for finding θ, ψ, and ϕ.

In matrix notation the above can be written as:

$$\begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} 1 & \sin\psi & 0 \\ 0 & \cos\phi\cos\psi & \sin\phi \\ 0 & -\sin\phi\cos\psi & \cos\phi \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}$$

In compact notation, represent the above equation by:

$$\Omega = M\,R$$

At this point we should digress somewhat to explain the term "orthogonal" matrix. In order to solve the above matrix equation for $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ we must find the inverse of the matrix which relates P, Q and R to the derivatives. The inverse is found from the equation

$$R = M^{-1}\Omega$$

which gives us an expression for $\dot{\psi}$, $\dot{\theta}$, and $\dot{\phi}$.

In the analysis which we did for the transformation from ground to body coordinate systems, we obtained the transformation matrix which related the body and the ground systems. This transformation matrix possesses a unique property: because of its nature, the inverse of this matrix is equal to its transpose. If we write the relationship between the ground and body systems as:

$$B = M(\psi,\theta,\phi)G,$$

then

$$G = M^{-1}(\psi,\theta,\phi)B.$$

However, for this matrix we can prove that

$$M^{-1}(\psi,\theta,\phi) = M^{T}(\psi,\theta,\phi).$$

When we can do this, the matrix, in this case the transformation matrix, is called orthogonal.

The matrix relating the angular rates in the body system, P, Q, R, with the derivatives of the Euler angles is not an orthogonal matrix. Therefore, we are compelled to derive its inverse. Without going into the actual computations of this, the inverse matrix is:

$$\begin{pmatrix}\dot\phi\\ \dot\theta\\ \dot\psi\end{pmatrix} = \frac{1}{\cos\psi}\begin{pmatrix}\cos\psi & -\sin\psi\cos\phi & \sin\psi\sin\phi\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi\cos\psi & \cos\phi\cos\psi\end{pmatrix}\begin{pmatrix}P\\ Q\\ R\end{pmatrix}$$

We are now in a position to know at any instant of time the angles ψ, θ, ϕ, merely by integrating their derivatives with respect to time. However, at this point we have merely replaced the three unknowns ψ, θ, ϕ by three other unknowns P, Q and R.

In body coordinates we can write the force vector as:

$$\vec{F} = F_1\vec{b}_1 + F_2\vec{b}_2 + F_3\vec{b}_3 = -mg\vec{g}_3 + \vec{F}_A + \vec{F}_T$$

We can solve for each of the components of the force in the body system by the simple expedient of taking the dot product with $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$, successively. Doing this we obtain the following three equations:

$$F_1 = -mg\vec{g}_3\cdot\vec{b}_1 + \vec{F}_A\cdot\vec{b}_1 + \vec{F}_T\cdot\vec{b}_1$$
$$F_2 = -mg\vec{g}_3\cdot\vec{b}_2 + \vec{F}_A\cdot\vec{b}_2 + \vec{F}_T\cdot\vec{b}_2$$
$$F_3 = -mg\vec{g}_3\cdot\vec{b}_3 + \vec{F}_A\cdot\vec{b}_3 + \vec{F}_T\cdot\vec{b}_3$$

Since we already have the transformation matrix between the ground and body system we can write down the trigonometric functions which relate the gravitational force, F_{gr}, to the body system. This gives us the series of equations:

$$F_1 = mg\cos\psi\sin\theta + \vec{F}_A\cdot\vec{b}_1 + \vec{F}_T\cdot\vec{b}_1$$
$$F_2 = -mg(\cos\phi\sin\psi\sin\theta + \sin\phi\cos\theta) + \vec{F}_A\cdot\vec{b}_2 + \vec{F}_T\cdot\vec{b}_2$$
$$F_3 = -mg(\cos\phi\cos\theta - \sin\phi\sin\psi\sin\theta) + \vec{F}_A\cdot\vec{b}_3 + \vec{F}_T\cdot\vec{b}_3$$

THE WIND SYSTEM

As yet we do not know what the transformation is between the aerodynamic force vector and the body system. We represented it above by the scalar product

between each unit vector in the body system, and the aerodynamic force vector, $\vec{F}_A$. As stated before, the aerodynamics are represented in the wind system, which is defined so that $\vec{w}_1$ always points parallel to the relative wind. We will define the total relative wind vector as $\vec{W}_w$. The vector is represented as:

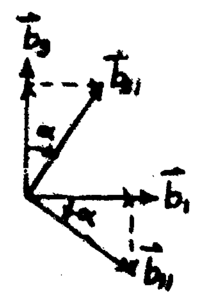
$$\vec{W}_w = W_w \vec{w}_1$$

where the "W_w" which is not a vector is the magnitude of the relative wind.

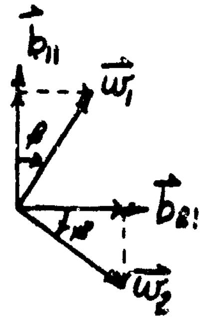
The aerodynamic forces arise entirely from the effect of the relative wind against the missile body. Therefore we need to represent the relative wind in a system which we can define. The force acting on the missile will be a force parallel to the direction of the relative wind. In order to perform this analysis we will transform the body system into the wind system by rotating through two angles, called the angles of attack. These angles are α and β. α is called the pitch angle of attack or, often, the angle of attack. β is called the yaw angle of attack or the sideslip angle. Collectively, we call α and β the angles of attack.

Previously when we were transforming from the ground to the body system, it was stated that three Euler angles were required in order to align the three ground axis vectors parallel to the corresponding body vectors. In the transformation from the body system to the wind system we are interested in aligning the $\vec{b}_1$ vector with the $\vec{w}_1$ vector only. To do this, we only need to rotate through two angles. The body two and body three vectors, of course, will be rotated at the same time, but, because we are interested in aligning only one vector, we will call the $\vec{w}_2$ and $\vec{w}_3$ unit vectors, those vectors which result from the $\vec{b}_2$ and $\vec{b}_3$ vectors after rotating the body system through the two angles of attack. This simplifies the derivation a great deal, since we needn't define a third angle.

The method of transformation is similar to the one used before. First we will rotate through a pitch angle of attack, that is, α , and then through the yaw angle of attack, β. Again we will observe our criterion for positive rotations. The following diagrams determine the transformations required to go from the body system to the wind system:



$$\vec{b}_{11} = \vec{b}_1 \cos\alpha - \vec{b}_3 \sin\alpha$$
$$\vec{b}_{21} = \vec{b}_2$$
$$\vec{b}_{31} = \vec{b}_1 \sin\alpha + \vec{b}_3 \cos\alpha$$



$$\vec{\omega}_1 = \vec{b}_{11}\cos\beta + \vec{b}_{21}\sin\beta$$
$$\vec{\omega}_2 = -\vec{b}_{11}\sin\beta + \vec{b}_{21}\cos\beta$$
$$\vec{\omega}_3 = \vec{b}_{31}$$

From these expressions we obtain a transformation matrix between the body and the wind system. This matrix (which is orthogonal) is:

$$\begin{pmatrix} \vec{\omega}_1 \\ \vec{\omega}_2 \\ \vec{\omega}_3 \end{pmatrix} = \begin{pmatrix} \cos\alpha\cos\beta & \sin\beta & -\sin\alpha\cos\beta \\ -\cos\alpha\sin\beta & \cos\beta & \sin\alpha\sin\beta \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix} \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix}$$

Utilizing the transformation matrix which we found during this rotation, we can now represent the aerodynamic forces, which are defined in the wind system, in terms of the body coordinate vectors. Thus we can write the following equation:

$$\vec{F}_A = D\vec{\omega}_1 + S\vec{\omega}_2 + L\vec{\omega}_3 = C\vec{b}_1 + Y\vec{b}_2 + N\vec{b}_3$$

The terms D, S and L in the wind system are given the following names: D is the drag, S is the side force, and L is the lift. These are the names which we apply to the aerodynamic forces in the specific direction shown above, when they are defined in the wind system. The terms in the body system are: C, the chord force, Y, the yaw force, and N, the normal force. These are the names given to the specific aerodynamic forces defined, as above, in the body coordinate system. If we now take the dot product with $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$, successively, we obtain expressions for the chord force, the yaw force, and the normal force in terms of the drag, side force, and lift. The dot products, in terms of the trigonometric functions of the angles of attack, can be immediately found from the transformation matrix which we just derived.

Derivation of Angles of Attack

So far we still seem to be behind. We may assume that the drag, side force, and lift are functions or curves which we know, but we have introduced two unknown quantities, the angles of attack. However, we can determine these angles of attack rather simply by means of the following derivation:

We can represent the relative wind vector as

$$\vec{W}_w = W_w \vec{w}_1 = W_1 \vec{b}_1 + W_2 \vec{b}_2 + W_3 \vec{b}_3$$

where W_1, W_2, and W_3 are the components of the relative wind defined in the body system. Taking the scalar product of the relative wind with $\vec{w}_1$ gives:

$$W_w = W_1 \vec{b}_1 \cdot \vec{w}_1 + W_2 \vec{b}_2 \cdot \vec{w}_1 + W_3 \vec{b}_3 \cdot \vec{w}_1$$

which, when we utilize the transformation matrix just defined, gives:

$$W_w = W_1 \cos\alpha \cos\beta + W_2 \sin\beta - W_3 \sin\alpha \cos\beta$$

The scalar product of the relative wind vector with $\vec{w}_2$ gives:

$$0 = W_1 \vec{b}_1 \cdot \vec{w}_2 + W_2 \vec{b}_2 \cdot \vec{w}_2 + W_3 \vec{b}_3 \cdot \vec{w}_2$$

or

$$0 = -W_1 \cos\alpha \sin\beta + W_2 \cos\beta + W_3 \sin\alpha \sin\beta$$

and the scalar product with $\vec{w}_3$ gives:

$$0 = W_1 \vec{b}_1 \cdot \vec{w}_3 + W_2 \vec{b}_2 \cdot \vec{w}_3 + W_3 \vec{b}_3 \cdot \vec{w}_3$$

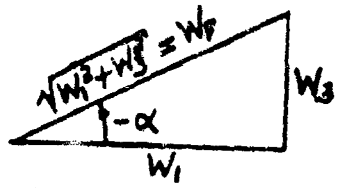
or

$$0 = W_1 \sin\alpha + W_3 \cos\alpha$$

Solving the last equation we obtain

$$-\tan\alpha = \tan(-\alpha) = \frac{W_3}{W_1}$$

We can now draw a triangle which gives us the relationship of a tangent of $(-\alpha)$:



From this we define sin $(-\alpha)$ and cos $(-\alpha)$, which are:

$$\sin(-\alpha) = -\sin\alpha = \frac{W_3}{W_P}$$

$$\cos(-\alpha) = \cos\alpha = \frac{W_1}{W_P}$$

If we now multiply the first of the previous equations by sin β, the second by cos β, and add, we obtain

$$\begin{array}{l} W_W \sin\beta = W_1 \cos\alpha \sin\beta \cos\beta + W_2 \sin^2\beta - W_3 \sin\alpha \sin\beta \cos\beta \\ \quad\;\; 0 = -W_1 \cos\alpha \sin\beta \cos\beta + W_2 \cos^2\beta + W_3 \sin\alpha \sin\beta \cos\beta \\ \hline W_W \sin\beta = W_2 \end{array}$$

or

$$\sin\beta = \frac{W_2}{W_W}$$

If we multiply the first equation by cos β, the second by sin β, and subtract, we obtain

$$\begin{array}{l} W_W \cos\beta = W_1 \cos\alpha \cos^2\beta + W_2 \sin\beta \cos\beta - W_3 \sin\alpha \cos^2\beta \\ \quad\;\; 0 = -W_1 \cos\alpha \sin^2\beta + W_2 \sin\beta \cos\beta + W_3 \sin\alpha \sin^2\beta \\ \hline W_W \cos\beta = W_1 \cos\alpha - W_3 \sin\alpha \end{array}$$

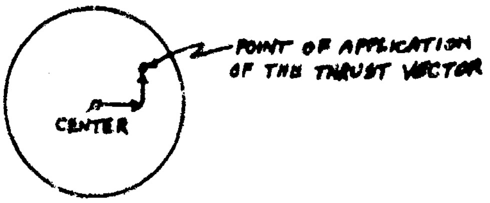
$$W_W \cos\beta = W_1 \left(\frac{W_1}{W_P}\right) - W_3 \left(\frac{-W_3}{W_P}\right) = \frac{W_1^2 + W_3^2}{W_P} = \frac{W_P^2}{W_P} = W_P$$

$$\therefore \cos\beta = \frac{W_P}{W_W}$$

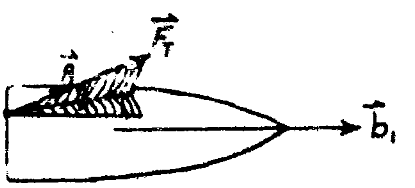
Thus, we have obtained exact expressions for the angles of attack, α, β, in terms of the components of the relative wind in the body system. The angles of attack are no longer unknowns; the terms of the relative wind are the new unknowns.

THE THRUST SYSTEM

The only force which has not been defined in the body system at this time is the thrust. Previously we said we were defining the thrust so that it was parallel with a $\vec{p}_1$ vector. It is necessary for the person who is doing a study with the missile to decide his own thrust misalignment. This means that the thrust vector (which always acts in the plane of the exit nozzle, not at the throat) can have a directional error (by error we mean it does not act along the missile center line) as well as an assumption that the thrust vector does not act at the center of the exit nozzle. Thus we can create an error in position as in the following diagram,



as well as a directional error as shown in the following diagram:



The analyst's discretion determines how large the thrust misalignment angles will be.* In general, thrust misalignment angles are, at most, a few degrees. Once we choose the misalignment angles, we can create a transformation matrix, as we did before, between the thrust axis system and the body system — by transforming the $\vec{b}_1$ vector into the $\vec{p}_1$ vector. As with the aerodynamics utilizing the angles of attack, we need only define two misalignment angles. If we call Γ_P the pitch and Γ_Y the yaw misalignment angles, and assume a transformation from the body to the thrust system by means of a Γ_P, Γ_Y sequence, the transformation matrix is precisely like that for the wind system with α replaced by Γ_P and β replaced by Γ_Y. Thus:

$$\begin{pmatrix} \vec{p}_1 \\ \vec{p}_2 \\ \vec{p}_3 \end{pmatrix} = \begin{pmatrix} \cos\Gamma_P \cos\Gamma_Y & \sin\Gamma_Y & -\sin\Gamma_P \cos\Gamma_Y \\ -\cos\Gamma_P \sin\Gamma_Y & \cos\Gamma_Y & \sin\Gamma_P \sin\Gamma_Y \\ \sin\Gamma_P & 0 & \cos\Gamma_P \end{pmatrix} \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix}$$

* In some cases it is possible that some sort of thrust misalignment angles will be furnished, but in general we will investigate a spectrum of values between some reasonable limits choosen by the analyst.

If the thrust misalignment angles were zero, of course, this implies (the more usual case) that the thrust acts along the missile center line and is parallel to the $\vec{b}_1$ vector. In any case, because of the way missiles are manufactured, we can assume that the thrust misalignment angles are very small, and utilize the "small angle" approximation. This gives the transformation matrix the form:

$$\begin{pmatrix} \vec{p}_1 \\ \vec{p}_2 \\ \vec{p}_3 \end{pmatrix} = \begin{pmatrix} 1 & \xi_Y & -\xi_P \\ -\xi_Y & 1 & 0 \\ \xi_P & 0 & 1 \end{pmatrix} \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix}$$

where we assumed $\xi_P \xi_Y = 0$.

If we presume that we have some sort of curve or function which represents the thrust characteristics of the actual missile with which we may be dealing, we can represent the thrust as:

$$\vec{F}_T = T\vec{p}_1 = T_1\vec{b}_1 + T_2\vec{b}_2 + T_3\vec{b}_3$$

Taking the scalar product with $\vec{b}_1$, $\vec{b}_2$, $\vec{b}_3$ sequentially, we obtain three equations for the magnitudes of the thrust parallel to the body coordinate axis:

$$T_1 = T\vec{p}_1 \cdot \vec{b}_1 = T$$

$$T_2 = T\vec{p}_1 \cdot \vec{b}_2 = T\xi_Y$$

$$T_3 = T\vec{p}_1 \cdot \vec{b}_3 = -T\xi_P$$

KINEMATICS AND RELATIVE WIND

At this point many unknowns have been solved, but it appears that as fast as we solve them we find new ones to take their places. Certainly this is true with the relative wind. Before we define the relative wind, which is a combination of the missile velocity with respect to the inertial frame and any external winds, we will investigate the missile velocity with respect to the ground system.

The Translation Accelerations

The differential equations describing the motion of a missile system must always be referred to the ground or inertial reference frame for solution. This is not meant to imply that every computation is done in the ground system — to the contrary, most computations are performed elsewhere. It does mean that the basic equations of motion must be referred to the inertial system and computations in other systems related to the inertial frame by appropriate transformation.

To develop the transformations between the inertial frame and rotating frame we are at liberty to choose a vector, assume it is in space, and define it in terms of its components in any coordinate system which we desire. But once we have chosen such a vector, subsequent differentiation and integration must involve the motions of the rotating frame.

When we describe a missile system, we are never likely to need range expressed in body coordinates, but we will need the range in the ground frame. We do need the velocity in body coordinates because the aerodynamics are based on missile velocity through air, so, in general, for our kinematics we wish to define motion by means of the velocity vector. Thus:

$$\vec{V}^{g} = \vec{V}^{b}$$

or

$$V_x \vec{g}_1 + V_Y \vec{g}_2 + V_z \vec{g}_3 = u\vec{b}_1 + v\vec{b}_2 + w\vec{b}_3$$

If we know the velocities in the body system then, simply finding the dot products with $\vec{g}_1$, $\vec{g}_2$, and $\vec{g}_3$, respectively, we can obtain the velocities in the ground system. The terms u, v, w are standard accepted symbols for the velocities in terms of the indicated body coordinate axes. The notation above, that is the superscripts "g" and "b", indicate in which coordinate system these vectors are defined. (Note that it is not always necessary to state in which coordinate system a vector is defined).

According to the above equations the velocity vector in the ground system is directly equal to the velocity vector in the body system. (We could have chosen the range and represented it in the same way, or we could have chosen the acceleration). In order to derive the kinematics, we will differentiate the velocity vector in the body system for the acceleration, integrate it in the body system, and then transform it to the ground system.

We previously found an expression relating the forces to the accelerations. Above we achieved an expression relating the velocities in the missile to the

velocities in the ground. We will now derive the accelerations in the body system. Newton's laws of motion are valid only in an inertial system, thus we must differentiate the expression for the velocity in the inertial system. That is:

$$\frac{{}^{g}d}{dt}\vec{V}^{g} = \frac{{}^{g}d}{dt}\vec{V}^{b}$$

It is immediately apparent that the expression on the right side of the equal sign is mixed and we can't work with it as it is. We define the derivative with respect to time in the inertial or ground system as the operator:

$$\frac{{}^{g}d}{dt} = \frac{{}^{b}d}{dt} + \vec{\omega}^{b} \times$$

so that

$$\frac{{}^{g}d}{dt}\vec{V}^{b} = \left(\frac{{}^{b}d}{dt} + \vec{\omega}^{b}\times\right)\vec{V}^{b} = \frac{d\vec{V}^{b}}{dt} + \vec{\omega}^{b}\times\vec{V}^{b}$$

It is now apparent why we went to such trouble, previously, to illustrate the equivalence between differentiation and cross product multiplication in a rotating coordinate system.

Completing the above differentiation gives:

$$\dot{\vec{V}} = \frac{{}^{b}d}{dt}(u\vec{b}_1 + v\vec{b}_2 + w\vec{b}_3) + \begin{vmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ P & Q & R \\ u & v & w \end{vmatrix}$$

or

$$\dot{\vec{V}} = \dot{u}\vec{b}_1 + \dot{v}\vec{b}_2 + \dot{w}\vec{b}_3 + (Qw - Rv)\vec{b}_1 + (Ru - Pw)\vec{b}_2 + (Pv - Qu)\vec{b}_3$$

and finally:

$$\dot{\vec{V}} = (\dot{u} + Qw - Rv)\vec{b}_1 + (\dot{v} + Ru - Pw)\vec{b}_2 + (\dot{w} + Pv - Qu)\vec{b}_3$$

Solving the force equation:

$$\Sigma \vec{F}^b = m\vec{A}^b = m\dot{\vec{V}}^b$$

or

$$\frac{F_1}{m}\vec{b}_1 + \frac{F_2}{m}\vec{b}_2 + \frac{F_3}{m}\vec{b}_3 = (\dot{u} + Qw - Rv)\vec{b}_1 + (\dot{v} + Ru - Pw)\vec{b}_2 + (\dot{w} + Pv - Qu)\vec{b}_3$$

This supplies us with the three translational equations of motion when we form the scalar products with $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$ respectively. Thus:

$$\dot{u} = \frac{F_1}{m} - Qw + Rv$$

$$\dot{v} = \frac{F_2}{m} - Ru + Pw$$

$$\dot{w} = \frac{F_3}{m} - Pv + Qu$$

The Rotational Accelerations

In the last section we developed three very pretty equations for the translational accelerations in the body system. However we still have the three unknowns, P, Q, and R which must be determined. P, Q, and R are also used for computing Ψ, Θ, and ϕ. These terms, which arise from the rotation of the body system, require three additional independent equations for their solution. We can find three such equations because the rotational accelerations are independent of the translational accelerations. As was done, in the last section, we find the rotational accelerations by solving Newton's second law of motion, in terms of the rotational accelerations.

We can write the equation for the torque as

$$\vec{T} = {}^{g}\frac{d}{dt}(I\vec{\omega})$$

where $\vec{T}$ is the torque, ${}^{g}d/dt$ means the derivative is taken in the ground coordinate system, and $(I\vec{\omega})$ is the angular momentum, where I is the moment of inertia tensor, and $\vec{\omega}$ is the angular velocity.

This rotational equation exactly parallels the translational equation previously defined.* I, the moment of inertia, corresponds to the mass of the translational equation, and $\vec{\omega}$, the rotational velocity, corresponds to the translational velocity. As pointed out in our previous discussion, in order to take the derivative in the ground system we must replace d/dt of the ground system by its equivalent operator in the body system. Thus:

$${}^{g}\frac{d}{dt}(I\vec{\omega}) = \left({}^{b}\frac{d}{dt} + \vec{\omega}\times\right)(I\vec{\omega})$$

which gives:

$${}^{g}\frac{d}{dt}(I\vec{\omega}) = {}^{b}\frac{d}{dt}(I\vec{\omega}) + \vec{\omega}\times(I\vec{\omega})$$

Note that in the above equation we have the term $\vec{\omega}\times(I\vec{\omega})$. This cannot be written as $I(\vec{\omega}\times\vec{\omega})$ (which is equal to zero) even though the inertia might be composed of constant terms, because I is a tensor. In general, "I" is represented as a matrix (or tensor) containing nine terms. Thus:

$$I = \begin{pmatrix} I_{11} & -I_{12} & -I_{13} \\ -I_{21} & I_{22} & -I_{23} \\ -I_{31} & -I_{32} & I_{33} \end{pmatrix}$$

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* The translational equation is: $\sum\vec{F} = {}^{g}d/dt(m\vec{V})$, that is, the time rate of change of the linear momentum is equal to the sum of the external forces.

However, in every rotating physical system, it can be proved that there exist three specific axes in which all terms except those on the main diagonal equal zero. Thus we can obtain the following matrix using a suitable transformation:

$$I = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix}$$

in which, if rotation occurred about these axes, the "cross product of inertia" would be zero. These axes are called the "principal axes of inertia". It is always possible to find such axes. Generally with a missile system, these axes must be found by experiment. However, if we take a homogeneous symmetrical body, then we can locate these principal axes of inertia by inspection. If we assume that our missile system is a symmetrical, homogeneous body, then the principal axes of inertia would lie along the same axes as the body system axes -- one of the primary reasons we chose to work in this system. Of course, the missile is not a homogeneous body, but for stability reasons, it is generally very close to being one. Almost always, therefore, we can assume, quite accurately, that the principal axes of inertia and the body axes coincide. If it were necessary to be more exact and actually use the complete form of the moment of inertia tensor, the easiest method for analysis would be to locate the principal axes of inertia, define the unit vectors parallel to them, and solve our rotational equations in that system. This would necessitate a transformation from the system in which the principal axes of inertia are located, to the body system. This involves a great deal of extra work (in the same way that transforming from a nonrotating system at the center of the earth to a rotating one on the surface of the earth requires extra work) and so we will just mention this fact but not go into details of calculations. If it ever becomes necessary for the analyst to deal with the general moment of inertia tensor, he must use the same concepts that we have been developing throughout this paper.

Assuming, now, that the principal axes of inertia and the body axes coincide, the term $I\vec{\omega}$ is:

$$I\vec{\omega} = \begin{pmatrix} I_{11} & 0 & 0 \\ 0 & I_{22} & 0 \\ 0 & 0 & I_{33} \end{pmatrix} \begin{pmatrix} P\vec{b}_1 \\ Q\vec{b}_2 \\ R\vec{b}_3 \end{pmatrix} = \begin{pmatrix} I_{11} P\vec{b}_1 \\ I_{22} Q\vec{b}_2 \\ I_{33} R\vec{b}_3 \end{pmatrix}$$

Furthermore, if the missile is shaped, roughly, like a cigar, rotation about $\vec{b}_2$ is indistinguishable from rotation about $\vec{b}_3$. That is:

the moment of inertia about $\vec{b}_3$, and that about $\vec{b}_2$, are substantially the same. Therefore, we can set $I_{22} = I_{33}$ and this gives us for $I\omega$:

$$I\vec{\omega} = I_{11}P\vec{b}_1 + IpQ\vec{b}_2 + IpR\vec{b}_3$$

where we have put

$$I_{22} = I_{33} = Ip.$$

We can now write the equation for the torque as:

$$\vec{T} = {}^{b}d/dt\ (I_{11}P\vec{b}_1 + IpQ\vec{b}_2 + IpR\vec{b}_3) + \begin{vmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ P & Q & R \\ I_{11}P & I_pQ & I_pR \end{vmatrix}$$

treating the moments of inertia as we did the mass (that is, we will extract the instantaneous value of the variable when we use it) we obtain:

$$\vec{T} = I_{11}\dot{P}\vec{b}_1 + \left[Ip\dot{Q}+RP(I_{11}-Ip)\right]\vec{b}_2 + \left[Ip\dot{R}+PQ(Ip-I_{11})\right]\vec{b}_3$$

If we define $I_{11} - I_p = I_s$, then the torque is:

$$\vec{T} = I_{11}\dot{P}\vec{b}_1 + (I_p\dot{Q} + RPI_s)\vec{b}_2 + (I_p\dot{R} - PQI_s)\vec{b}_3$$

Another name for the torque is "moment of momentum". A physical law states "the time rate of change of the angular momentum (the moment of momentum) is equal to the sum of the external moments". In equation form this is:

$${}^{s}d/dt\ (I\vec{\omega}) = \Sigma\vec{M}$$

or component wise

$$\vec{T} = {}^{s}d/dt\ (I\vec{\omega}) = \Sigma\vec{M} = \Sigma L\vec{b}_1 + \Sigma M\vec{b}_2 + \Sigma N\vec{b}_3$$

where ΣL, ΣM, ΣN are standard notation for the external moment compodents in the indicated body directions. Taking the dot product of the components of the torque with $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$, we obtain

$$I_{11}\dot{P} = \Sigma L$$

$$I_p\dot{Q} + PRI_s = \Sigma M$$

$$I_p\dot{R} - PQI_s = \Sigma N$$

Thus we can write the following three equations for the angular acceleration:

$$\dot{P} = \frac{\Sigma L}{I_{11}}$$

$$\dot{Q} = \frac{\Sigma M - PRI_s}{I_p}$$

$$\dot{R} = \frac{\Sigma N + PQI_s}{I_p}$$

If we know the external moments in the terms ΣL, ΣM, and ΣN, we can solve for P, Q, and R, by integrating the respective angular accelerations.

It appears that we have substituted the unknowns, ΣL, ΣM and ΣN for the other unknowns P, Q, and R, but we can determine the external moments by physical experiments, whereas it may be difficult to get the angular rates.

We now have six aerodynamic quantities which must be evaluated. From the translational equations we needed three aerodynamic terms D, S, and L. Now from the rotational equations we have three more aerodynamic terms contained in ΣL, ΣM, and ΣN. We will discuss these shortly.

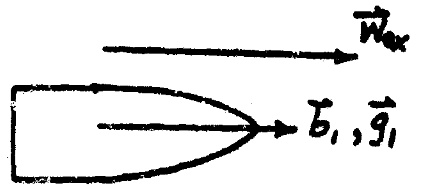
<u>Relative and External Wind</u>

From some source (as for example, air-weather at WSMR) we need to obtain figures for the external wind. The values which we get for the external wind will be given to us in terms of directions such as north or east, etc. From the given direction we must obtain the components along each of our ground axes.

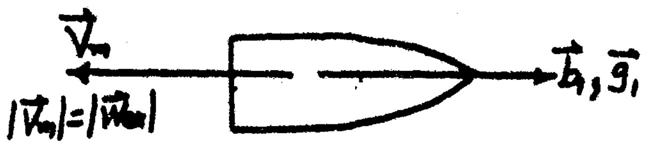
Once we have obtained these figures in the ground system we can represent the external wind as in the following equation:

$$\vec{W}_{ex} = W_x \vec{g}_1 + W_y \vec{g}_2 + W_z \vec{g}_3$$

The method for determining the relative wind can be easily seen from the following analysis. Suppose we have an object sitting at rest on the ground



so that the body-1 axis is parallel to the ground-1 axis and the external wind is blowing parallel to the ground-1 axis. Then, as far as the missile, itself, is concerned, the wind acts on it as if the air were still and the missile were flying backwards, with a velocity equal in magnitude to that of the external wind. This can be seen from the following diagram:



Thus we would write, for the relative wind illustrated above:

$$\vec{V}_r = -|\vec{W}_{ex}|\, \vec{g}_1$$

That is, if we know the direction in which the external wind is blowing, then the effect on the missile is merely the negative value of this quantity.

In our original equation for the external wind, we assumed that the symbols for the components were positive numbers. The sign of the components is determined by the direction the air is moving.

The first thing we must do with the external wind vector is transform it into the body system. That is:

$$\vec{W}_{ex} = W_x\vec{g}_1 + W_y\vec{g}_2 + W_z\vec{g}_3 = W_{ex1}\vec{b}_1 + W_{ex2}\vec{b}_2 + W_{ex3}\vec{b}_3$$

The components of the external wind in the body system are found by taking the dot product with $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$, respectively, and utilizing the transformation matrix. This gives:

$$W_{ex1} = W_x\vec{g}_1\cdot\vec{b}_1 + W_y\vec{g}_2\cdot\vec{b}_1 + W_z\vec{g}_3\cdot\vec{b}_1$$

$$W_{ex1} = W_x\cos\psi\cos\theta + W_y(-\cos\phi\sin\psi\cos\theta + \sin\phi\sin\theta) + W_z(\sin\phi\sin\psi\cos\theta + \cos\phi\sin\theta)$$

$$W_{ex2} = W_x\vec{g}_1\cdot\vec{b}_2 + W_y\vec{g}_2\cdot\vec{b}_2 + W_z\vec{g}_3\cdot\vec{b}_2$$

$$W_{ex2} = W_x\sin\psi + W_y\cos\phi\cos\psi - W_z\sin\phi\cos\psi$$

$$W_{ex3} = W_x\vec{g}_1\cdot\vec{b}_3 + W_y\vec{g}_2\cdot\vec{b}_3 + W_z\vec{g}_3\cdot\vec{b}_3$$

$$W_{ex3} = -W_x\cos\psi\sin\theta + W_y(\cos\phi\sin\psi\sin\theta + \sin\phi\cos\theta) + W_z(-\sin\phi\sin\psi\sin\theta + \cos\phi\cos\theta)$$

The velocity of the missile is:

$$\vec{V}_m = u\vec{b}_1 + v\vec{b}_2 + w\vec{b}_3$$

Noting, that the relative wind, $\vec{W}_r$, is the velocity of the missile minus the external wind, we have:

$$\vec{W}_r = W_1\vec{b}_1 + W_2\vec{b}_2 + W_3\vec{b}_3$$

where

$$W_1 = u - W_{ex1}$$

$$W_2 = v - W_{ex2}$$

$$W_3 = w - W_{ex3}$$

and we have completely defined the relative wind.

EXTERNAL FORCES AND MOMENTS

At this point we have derived six independent equations describing the missile's motion:

$$\dot{u} = F_1/m + Rv - Qw$$

$$\dot{v} = F_2/m + Pw - Ru$$

$$\dot{w} = F_3/m + Qu - Pv$$

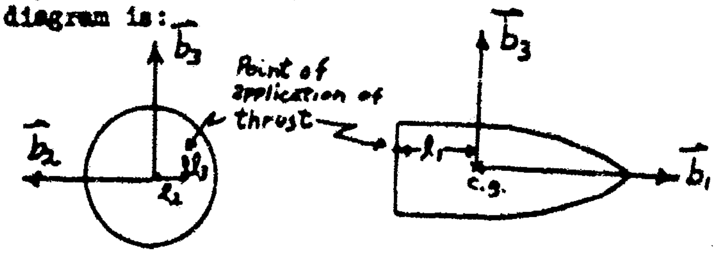
$$\dot{P} = \Sigma L/I_{11}$$

$$\dot{Q} = (\Sigma M - PRI_s)\frac{1}{I_p}$$

$$\dot{R} = (\Sigma N + PQI_s)\frac{1}{I_p}$$

The mass, m, and the moments of inertia are functions or curves which are obtained, either from the manufacturer or from derivations of data obtainable from the manufacturer. Assuming that these curves are available, the unknowns in our equations, as yet, are the external moments for the rotational equations and the external forces for the translational equations. The external forces arise from three sources: gravity, aerodynamics, and thrust. The gravitational and thrust forces were discussed earlier, so the aerodynamic forces are the only ones which we need to discuss in greater detail. The external moments are of two types: aerodynamic moments and moments caused by thrust misalignment.

Earlier we defined the thrust axis system. External moments will arise in the body system due to the position and direction of the thrust vector. This earlier diagram is:



There are three components of the thrust acting in the body system. These are found by transforming the thrust vector (as was done before) into the body system. This gives us the equation:

$$\vec{T} = T_1\vec{b}_1 + T_2\vec{b}_2 + T_3\vec{b}_3.$$

Presuming that the point of application is as pictured above, the vector from the origin of the body system to the point of application of the thrust vector is:

$$\vec{\ell} = -\ell_1\vec{b}_1 - \ell_2\vec{b}_2 + \ell_3\vec{b}_3$$

The vector equation for the moment due to thrust misalignment is:

$$\vec{M}_{TM} = \vec{\ell} \times \vec{T},$$

which gives:

$$\vec{\ell} \times \vec{T} = \begin{vmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \\ -\ell_1 & -\ell_2 & \ell_3 \\ T_1 & T_2 & T_3 \end{vmatrix} = (-\ell_2 T_3 - \ell_3 T_2)\vec{b}_1 + (\ell_3 T_1 + \ell_1 T_3)\vec{b}_2 + (-\ell_1 T_2 + \ell_2 T_1)\vec{b}_3$$

We can write the three terms for the external moments arising from thrust misalignment as:

$$\vec{M}_{TM} = L_T \vec{b}_1 + M_T \vec{b}_2 + N_T \vec{b}_3$$

Then the sum of the external moments is:

$$\Sigma L = L_A + L_T$$

$$\Sigma M = M_A + M_T$$

$$\Sigma N = N_A + N_T$$

where the subscript "A" implies aerodynamic moments.

The external forces can be written as:

$$F_1 = F_{1A} + T_1 + F_{1gr}$$

$$F_2 = F_{2A} + T_2 + F_{2gr}$$

$$F_3 = F_{3A} + T_3 + F_{3gr}$$

and, from both sets of equation, only the aerodynamic functions have not been completely defined.

Representation of Aerodynamics

It has been found, after many years of experience, that the most convenient method for representing aerodynamic forces and moments is:

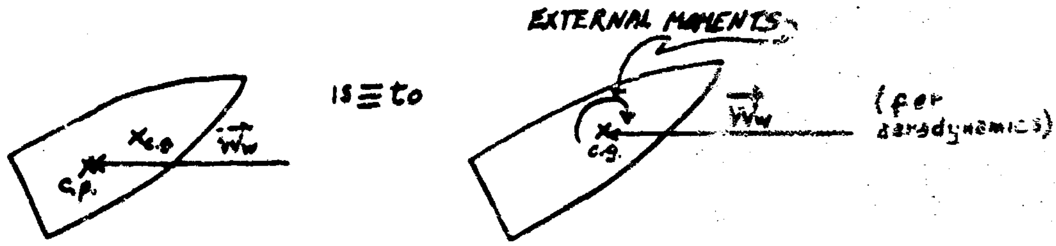
$$F_r = \frac{1}{2}\rho W_w^2 S C_r = q S C_r = \frac{1}{2} p \gamma M^2 S C_r$$

$$M_\Lambda = \frac{1}{2}\rho W_w^2 S \bar{c} C_\Lambda = q S \bar{c} C_\Lambda = \frac{1}{2} p \gamma M^2 S \bar{c} C_\Lambda$$

where F_r is an arbitrary aerodynamic force, M_Λ an arbitrary aerodynamic moment, ρ the density of the air, p the air pressure, S and $\bar{c}$ "characteristic"

surface and length, respectively;* γ the ratio of the heat capacities of the medium: $\gamma = C_p/C_v$ -- which for air is about 1.4, M is Mach number, W_w the relative wind, C_F and C_A the force and moment coefficients, and q is the dynamic pressure.

Many times in the past (and still fairly often) we were given the aerodynamic forces, and a position on the missile called center of pressure. The center of pressure is the point at which the aerodynamic forces act on the missile. Since the missile in space is a free-free beam, we can represent the moments and forces, independently, as acting at the center of gravity. The sum of the external forces acts through the c.g., and the sum of the external moments act as torques around the c.g. One point of view is equivalent to the other. This can be observed from the following diagram:



The equations, as defined earlier, assume that we have forces acting through the center-of-gravity and that we have moments tending to rotate the missile about the c.g.

The air density, ρ , in the force and moment equations, is determined from a "standard atmosphere". We treat the points on the curve for ρ as we did the points on the curves for mass, moment of inertia and thrust. From the above equations, then, only Mach number and the coefficients are left undefined. Mach number is defined as:

$$M = \frac{W_w}{V_s}$$

where V_s is the local speed of sound. V_s can be represented as a function of altitude. It is generally a "standard" curve, as is ρ, the air density.

The three force coefficients are defined in the wind system as follows:

C_D = drag force coefficient

C_S = side force coefficient

C_L = lift force coefficient

* Generally, S is the area of the cross section of the missile having the greatest diameter and $\bar{c}$ is the "mean aerodynamic chord."

or if they are defined in the body system they are named:

C_C = chord force coefficient

C_Y = yaw force coefficient

C_N = normal force coefficient

The moment coefficients, defined in the body system, are:

C_ℓ = rolling moment coefficient

C_m = pitching moment coefficient

C_n = yawing moment coefficient

Notice that the subscripts are capital letters for forces and lower case letters for moments.

It has been found in most cases that we can represent the aerodynamics as a sum of coefficients, each of which is a function of two variables, one of them Mach number. Instead of Mach number, we sometimes use Reynold's number for low velocities, and we may use a parameter called Froude's number for very high altitudes where the air cannot be considered as a continuous medium. But, as far as we are concerned, our independent "number" will be Mach number. For most purposes, just a few variables as functions of Mach number, will suffice to describe the aerodynamic forces. For example, α, the angle of attack in pitch, will sometimes represent as high as 80% of the total aerodynamic pitching force or moment. β, the sideslip angle or yawing angle of attack, will do the same for the yaw moment or yaw force. The second most important parameter (from the standpoint of missile control it is the <u>most</u> important) is the fin deflection, δ. In addition to the two parameters mentioned above, that is angles of attack and fin deflections, other parameters often used are the square of the angles of attack, the time rate of change of the angles of attack, the pitching, yawing, and rolling rates, and combinations of, say, the angles of attack and fin deflection -- and other interdependent parameters.

In most representations of missile systems we "linearize" the aerodynamics. We can write the aerodynamic coefficient, say C_m, as a function of its various parameters:

$$C_m = C_m(\alpha, \delta, \alpha^2, \dot{\alpha}, Q, \ldots)$$

Taking the derivative of this function and assuming independence of each variable, we obtain:

$$\frac{dC_m}{dM} = \frac{\partial C_m}{\partial \alpha}\frac{d\alpha}{dM} + \frac{\partial C_m}{\partial \delta}\frac{d\delta}{dM} + \frac{\partial C_m}{\partial \alpha^2}\frac{d\alpha^2}{dM} + \frac{\partial C_m}{\partial \dot{\alpha}}\frac{d\dot{\alpha}}{dM} + \frac{\partial C_m}{\partial Q}\frac{dQ}{dM} + \ldots$$

In standard usage the notation is:

$$\frac{\partial C_m}{\partial \alpha} = C_{m\alpha}, \quad \frac{\partial C_m}{\partial \delta} = C_{m\delta}, \text{ etc.}$$

and we can write:

$$\frac{dC_m}{dM} = C_{m\alpha}\frac{d\alpha}{dM} + C_{m\delta}\frac{d\delta}{dM} + C_{m\alpha^2}\frac{d\alpha^2}{dM} + C_{m\dot{\alpha}}\frac{d\dot{\alpha}}{dM} + C_{mQ}\frac{dQ}{dM} + \ldots$$

If we now shift the burden of variation of the term due to change in Mach number to the coefficients, we can write the above as:

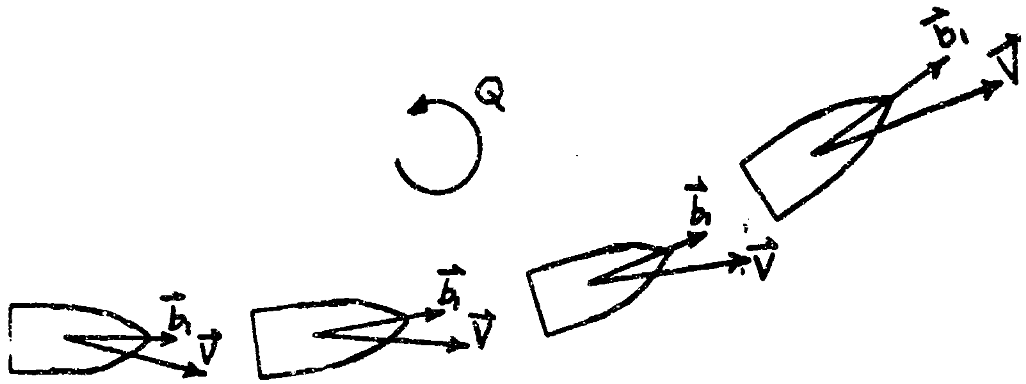
$$dC_m(M) = C_{m\alpha}(M)\,d\alpha + C_{m\delta}(M)\,d\delta + C_{m\alpha^2}(M)\,d\alpha^2 + C_{m\dot{\alpha}}(M)\,d\dot{\alpha} + C_{mQ}(M)\,dQ + \ldots$$

Integrating this expression gives:

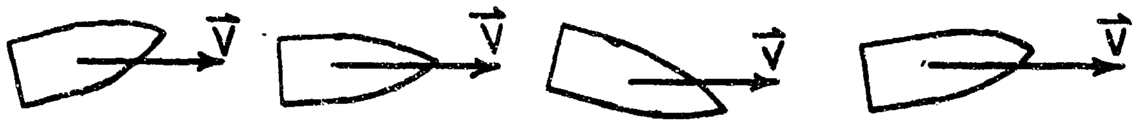
$$C_m = C_{m\alpha}\alpha + C_{m\delta}\delta + C_{m\alpha^2}\alpha^2 + C_{m\dot{\alpha}}\dot{\alpha} + C_{mQ}Q + \ldots$$

which gives us a linearized term for the pitching moment. Although the above equation looks very beautiful, and is generally quite accurate, it is not exact. This is because a coefficient is not necessarily independent of the separate variables. To the contrary. A change in, say δ, will perhaps affect $C_{m\alpha}$. That is, the value of the coefficient $C_{m\alpha}$ is not the same for different values of δ. Therefore, if we want greater accuracy, we would have to include equations of the form: $C_{m\alpha\delta}\alpha\delta$, etc. As was said before, we are generally more accurate if we assume independence of the different parameters than the actual accuracy of the aerodynamic coefficients, themselves. The above parameters (which, in the example, were shown as functions of the pitching moment) are, in general, the most important aerodynamic parameters.

At this point let us digress and show that $\acute{\alpha}$ is not necessarily the same as Q. The following diagram illustrates a change in Q with no change in α :

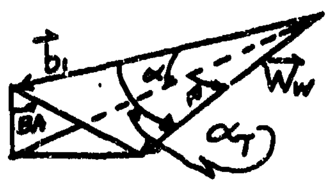


On the other hand, the following diagram



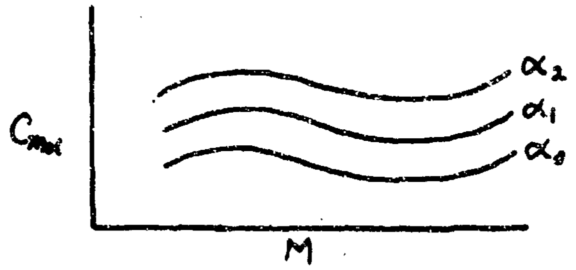
shows a change in α that is equal to Q.

With the advent of the last series of equations, presuming that the manufacturer or designer of the particular vehicle under study has provided functions for the aerodynamics, then all the unknowns, which began with the original force equation, are now known except for δ. The aerodynamic functions furnished by the manufacturer will probably not be in exactly the form shown. For example, some missiles only require such aerodynamic functions are, say, α, δ, and Q. Other manufacturers bring in completely different parameters. For example, the aerodynamics of one missile use the "bank angle". This is an angle which lies between the pitch angle of attack and the hypotenuse generated by the side slip angle and the pitch angle of attack. This is illustrated in the following diagram:



Notice that the "hypotenuse" is actually the total angle of attack, α_T. Whatever form the aerodynamics are represented in, their relationships are of the form shown above.

As some times happens, linearized representation of the aerodynamics is not sufficient. (Repeatedly, throughout this paper, attention has been called to possible sources of error, approximations used, etc. For example, it was pointed out that the surface of the earth is not actually an inertial system nor is it flat; and that we had to find a set of axes, other than the body axes, to correspond exactly to the principal axes of inertia). Generally, our error in linearization is not very great. If it is, then we must find other methods for determining the aerodynamics. If, for example, we represent $C_{m\alpha}$ as a curve, and if it is linearized, it will be a plot somewhat as follows:



where M is Mach number. If, however, $C_{m\alpha}$ were also a function of, say, ϕ, the roll angle, then our curve would have to be "three dimensional", so that by moving in the "third dimension" (corresponding to a change of ϕ) we would find the desired coefficient as a point on the surface.

TRAJECTORY INFORMATION

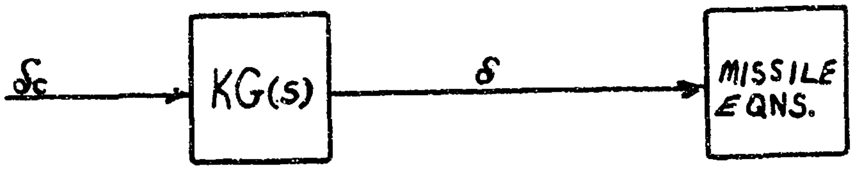
The ultimate purpose of an analysis, such as this, is to determine the trajectory of the missile under the influence of the various parameters. As was stated earlier, we made our velocity vector the invariant between the inertial system and the rotating, or body system. As a general rule, trajectory information demands the range vectors as a function of time. We wish to know the ranges, but we wish to know these ranges in the ground system. Therefore, we will integrate the accelerations in the body system, that is, $\dot{u}$, $\dot{v}$, $\dot{w}$ and obtain u, v, w, the velocities, and transform them into the ground system. These are:

$$V_x \vec{g}_1 + V_y \vec{g}_2 + V_z \vec{g}_3 = \vec{V} = u\vec{b}_1 + v\vec{b}_2 + w\vec{b}_3$$

V_x, V_y and V_z are found by forming the dot product with $\vec{g}_1$, $\vec{g}_2$, $\vec{g}_3$, respectively and utilizing the transformation matrix. To obtain the three ranges in the ground system, we now integrate these three components of velocity. This gives us our ground range. If we keep track of the elapsed time during computation, we have a representation of the ranges in the ground system at any instant of time.

FIN DEFLECTIONS

Assuming that the analysis just developed is for a missile with guidance, then there is some form of external "guidance signal" which is an input to the guidance circuitry. Let us represent the "guidance signal" or "guidance command" by δ_c. δ_c may be a radar signal, a doppler shift, signals from a stable platform, etc. The "fin deflection" may not, literally be an angle, but may be a proportional angle, a spoiler effect, a dwell-time, etc., but in any case we can represent the relationship of the "delta command," δ_c, to the "delta achieved", δ, by means of a transfer function. Thus:



With δ_c, as yet, our only unknown, we can represent the entire missile airframe (with the guidance represented as δ_c KG(s) = δ) by means of block diagrams. These block diagrams help us to visualize the interrelationships and interdependences of the equations which we have been deriving. On the following page is a block diagram of our system of equations.

The mathematics for every missile system will be basically similar to the mathematics we have derived. This system, described, is far from being complete. As yet we have not mentioned guidance, which will involve such things as seekers, targets, radars, preprograms, etc. What we have done, however, is mathematically describe the missile dynamics, and indicate methods for more exact analysis if needed.

TRANSFER FUNCTIONS

In the previous analysis we had but one unknown, the fin deflection, δ. The "fin deflection" represents either the actual fin position as an angle, or a "dummy" parameter representative of the fin deflection. If δ_c is the command to the missile, we can then represent the process between δ_c and δ as a transfer function, KG(s). A transfer function is merely a Laplace transform of the hardware in the missile with zero initial conditions. Transfer functions, as such, will not be taken up in this paper. Suffice to say, there exist a number of good texts on the Laplace transform and transfer functions*.

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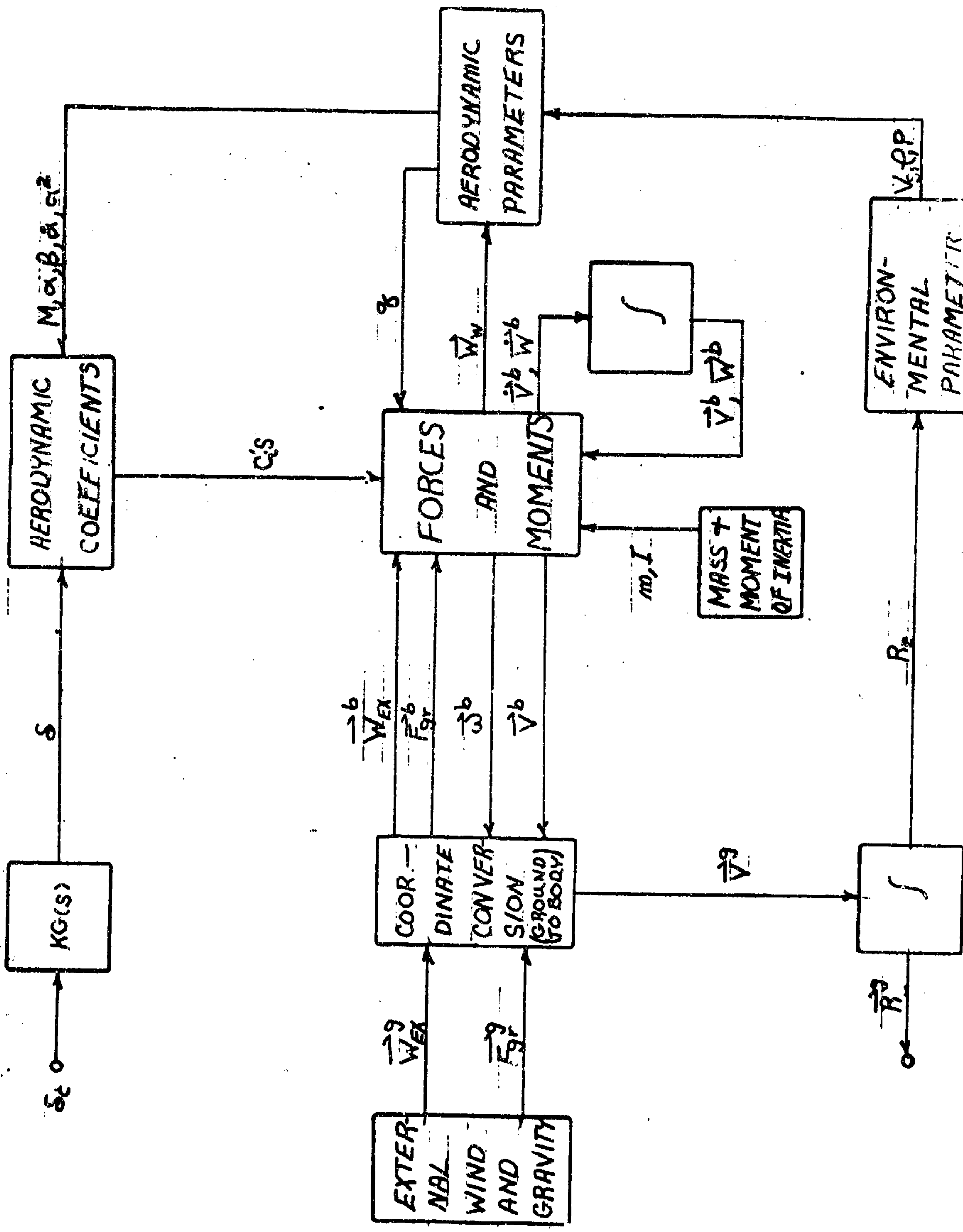
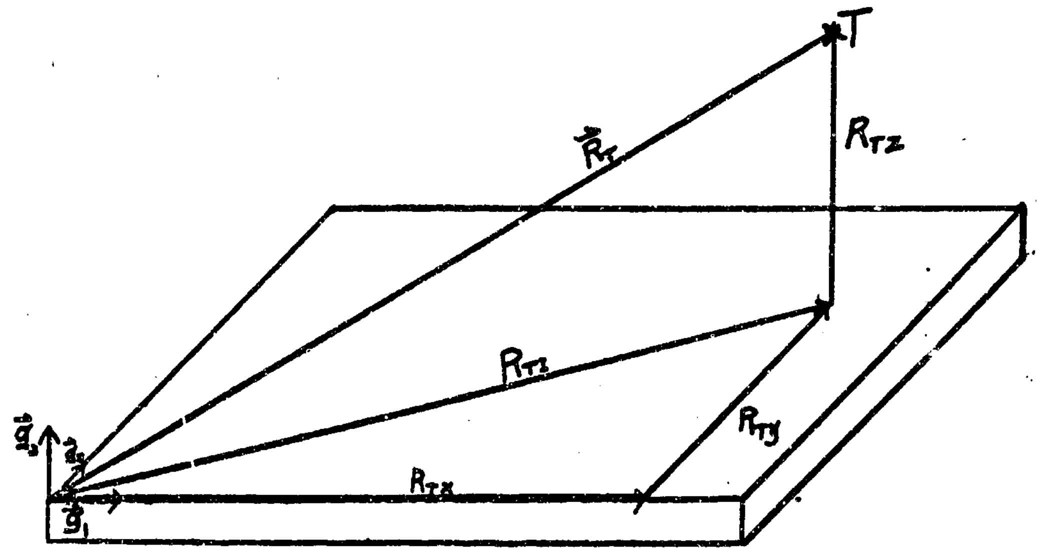
* For example, see: Thaler, G. S. and Brown, R. G., Servomechanism Analysis, McGraw-Hill, 1953.



MATHEMATICAL MODEL BLOCK DIAGRAM

TARGETS

For any type of problem in which a target is used, the target position and velocity will be represented in ground coordinates. The following diagram illustrates the target range vector and its components along the three ground axes:



Thus, the target range vector is:

$$\vec{R}_T = R_{Tx}\vec{g}_1 + R_{Ty}\vec{g}_2 + R_{Tz}\vec{g}_3$$

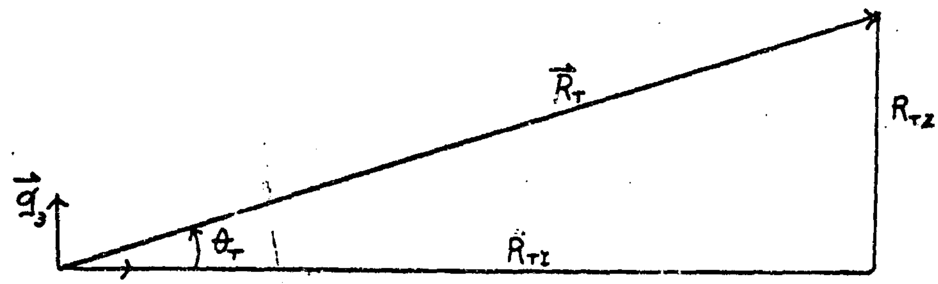
If we also know the target velocity we can represent it as:

$$\vec{V}_T = V_{Tx}\vec{g}_1 + V_{Ty}\vec{g}_2 + V_{Tz}\vec{g}_3$$

The coordinate system fixed to the surface of the earth (whether it is assumed to be the inertial system or whether it is rotating and the center of the earth is taken as the inertial system) represents a convenient way (in fact, probably the only way) which we can introduce target ranges and velocities into our system of equations.

At times the analyst is allowed to be fairly arbitrary in his choice of ranges and velocities of the target -- insofar as these choices represent values which can be realized. Many times it is convenient to have a target with a velocity along one of the ground axes but no velocity component along any other axis. This eases the complexity of the problem. When this situation arises (that is, when a target has a constant velocity, perhaps, or even a constant range) it is often desirable to orient the ground system so that the target "flies" parallel to one of the unit vectors.

The method for obtaining the individual components of the target range and velocity will not be the same as we have discussed before. Previously we transformed from one coordinate system to another by means of an Euler angle sequence. This method was used because we wished to rotate one set of axes (in this case, the ground axes) so that their final position would be aligned with another set of axes (in this case, the body axes). We can assume that the target is a point in space. Therefore, we have a vector which, if it is the range vector, reaches from the origin of a ground system to a point which we call the target. This vector is then represented as having three components parallel to the ground axis system. In order to determine the components of the target range (or velocity) we will use a method called "rectangular projection". First we obtain a projection of the vector in the $\vec{g}_1 - \vec{g}_2$ plane, an intermediate range vector, $\vec{R}_{TI}$, as in the above diagram. This length is found by rotating through the target elevation angle, θ_T, as in the following diagram:

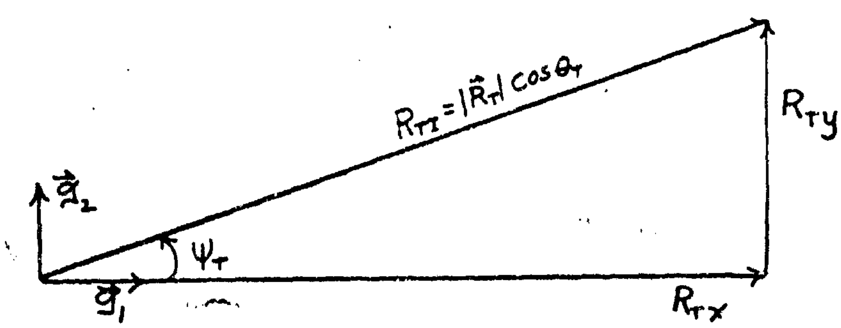


which gives for R_{TZ} and R_{TI} the values:

$$R_{TI} = |\vec{R}_T| \cos\theta_T$$

$$R_{TZ} = |\vec{R}_T| \sin\theta_T$$

The following diagram illustrates this new component of the original vector:



If we introduce a target azimuth angle, ψ_T , then the other two components of the target, in the $\vec{g}_1 - \vec{g}_2$ plane are:

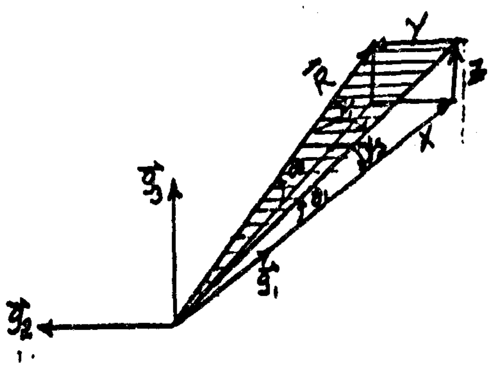
$$R_{Tx} = |\vec{R}_T| \cos\theta_T \cos\psi_T$$

$$R_{Ty} = |\vec{R}_T| \cos\theta_T \sin\psi_T$$

and we have completely represented the target position vector.

The projection, as defined above, assumes that our intermediate vector is projected onto the horizontal plane, that is, the $\vec{g}_1$ - $\vec{g}_2$ plane. One may wonder if it wouldn't be as correct to project the original range vector onto a vertical plane such as the $\vec{g}_1$ - $\vec{g}_3$ or $\vec{g}_2$ - $\vec{g}_3$ plane, and, from the intermediate vector so projected, get the components of the range vector. Mathematically and geometrically, either method is equivalent. The first method corresponds to a θ projection followed by a ψ projection; the second to a ψ, θ projection.

The limitations imposed on the choice of the projection depends on which angles are measured. In the above instance, ψ_T and θ_T, are angles of a type which a radar system might measure. The vertical angle, θ_T, is measured by a radar from the horizontal plane, that is, the $\vec{g}_1$ - $\vec{g}_2$ plane, and the horizontal angle, ψ_T, is measured in this same horizontal plane. In such a situation, we are forced to use a vertical projection first. This can be demonstrated by reference to the following figure:



If we project first onto the X-Z plane through the angle ψ_1, then for the azimuth angle we have:

$$\sin\psi_1 = \frac{Y}{|R|}$$

If we project first onto the X-Y plane, then the sine of the azimuth angle, ψ_2 is:

$$\sin\psi_2 = \frac{Y}{|R|\cos\theta_2}$$

Furthermore, projecting first on the X-Z plane gives

$$\sin\theta_1 = \frac{Z}{|R|\cos\psi_1}$$

for the elevation angle, and

$$\sin\theta_2 = \frac{Z}{|R|}$$

if we first project the range vector onto the horizontal plane.

Obviously our projection method depends upon which angles were measured. Radar angles necessarily proceed through a Θ projection followed by a Ψ.

At this point it is difficult to say what will be done with the target range and the target velocities once they are expressed in the ground coordinate system. Each kind of missile has a different guidance philosophy and will, therefore, utilize these parameters in different ways. But, in every case, the guidance philosophy must use them in one way or another. We will not attempt to discuss guidance now, but will illustrate some types later. Suffice to say, we can transform the target velocity vector into our missile body coordinates if we wish; we can use the target range for a radar ground station; or we can utilize both these vectors in several different ways. If, perhaps, the guidance of the missile needs the distance of the missile to the target, then we can find the relative distance between target and missile simply by subtracting the components in the ground system and then, if we wish to express these relative components in missile body coordinates, we can transform them immediately with the aid of the transformation matrix.

As a closing paragraph to this subject we might mention two situations in which we would need to transform these vectors. For example, if the guidance of the missile depends upon the intensity of some source of radiation relative to a guidance head in the missile, such as infra-red radiation, then the brightness of the radiation at the surface of the detector would depend upon the distance of the missile from the source. In this case we would like to know how far the missile is from the source. After finding this distance, we would need to transform it into missile coordinates so that we could find the relationship between the direction of the detector and the missile heading direction — and determine tracking error, capability, etc. As another example, the maneuvers of the missile might depend upon the relative velocity of the target with respect to the missile. In this case we would find the relative velocity between the target and missile and transform the resultant components into the body system. This would give desired information to the missile.

GYROS

Before beginning the discussion of gyros we would like to insert at this point a demonstration of a statement which we will need later in the discussion. We recall that earlier we derived a transformation matrix between the ground and body systems. This particular matrix was called an "orthogonal" matrix. One property of this matrix, which is important in the following discussion, is that each element of the matrix is equal to its co-factor. For example, a matrix using the Ψ, ϕ, and Θ, is:

$$\begin{pmatrix} \vec{\mathfrak{g}}_1 \\ \vec{\mathfrak{g}}_2 \\ \vec{\mathfrak{g}}_3 \end{pmatrix} = \begin{pmatrix} -\sin\Theta\sin\phi\sin\Psi + \cos\Theta\cos\Psi & \sin\Theta\sin\phi\cos\Psi + \cos\Theta\sin\Psi & -\sin\Theta\cos\phi \\ -\cos\phi\sin\Psi & \cos\phi\cos\Psi & \sin\phi \\ \cos\Theta\sin\phi\sin\Psi + \sin\Theta\cos\Psi & -\cos\Theta\sin\phi\cos\Psi + \sin\Theta\sin\Psi & \cos\Theta\cos\phi \end{pmatrix} \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix}$$

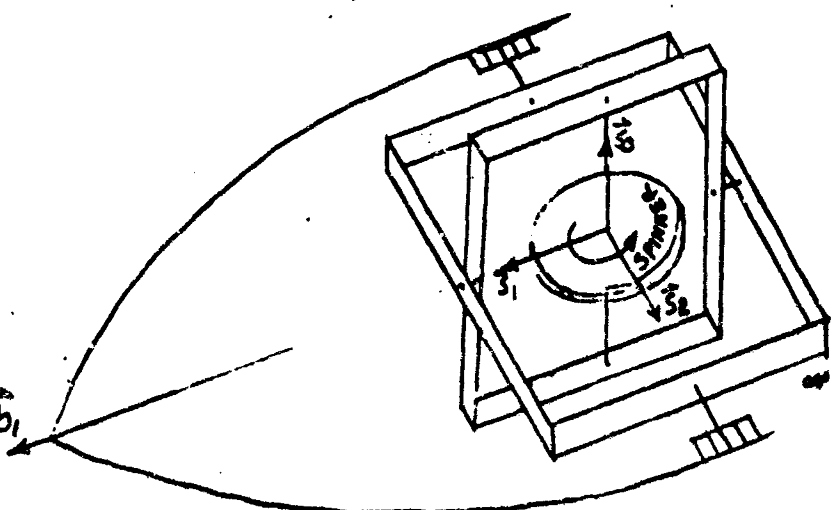
The term in the upper left-hand corner of the above matrix has, as its co-factor, the determinant found by deleting the first row and first column. If we solve this determinant we get the following term:

$$\begin{vmatrix} \cos\phi\cos\psi & \sin\phi \\ -\cos\theta\sin\phi\cos\psi+\sin\theta\sin\psi & \cos\theta\cos\phi \end{vmatrix} = \cos^2\phi\cos\theta\cos\psi+\sin^2\phi\cos\theta\cos\psi - \sin\theta\sin\psi\sin\phi$$

$$= \cos\theta\cos\psi - \sin\theta\sin\psi\sin\phi$$

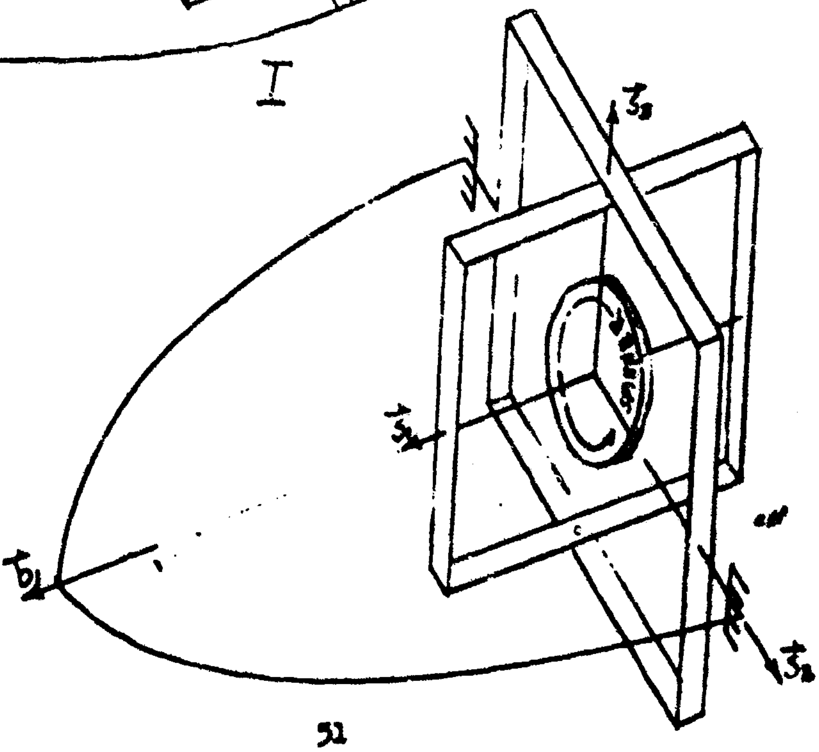
which is the first term in the matrix. This same type of operation can be shown for every element of the matrix.

In a missile which obtains its position by means of gyros, generally two two-degree-of-freedom gyros are employed. These gyros are "slaved" in order to measure all of the angles. The following two diagrams illustrate this. Each diagram shows a schematic of a gyroscope having a pitch outside gimbal.



I

II



Gyroscopes, of course, may have fixed outside gimbals other than pitch. There are six possibilities in all: two pitch, two yaw, and two roll. The second angle further differentiates between types of gyros, there being two possibilities for each type, as is illustrated in the above diagrams.

At the center of each gyroscope is a spinning mass whose axes of rotation remain fixed in space. If the initial direction of these axes is known with respect to the ground coordinates, then, within the missile itself, is a vector which retains a fixed direction in ground coordinates. The fixed axis is spinning very rapidly, but by integration of the spinner angular velocity an angle about this axis can be determined. We will not actually use this angle because it would not be accurate. We will use only pick-off angles of the two outer gimbal rings. We can see that a sequence of angles from the internost spinning mass to the outer gimbal ring, which is fixed to the missile, will give a sequence of angles which determine the missile position in space with respect to the ground coordinate system.

We cannot "arbitrarily" choose the Euler angle sequence of the angles in the gyroscope as we did in the transformation between the ground and the body. Once the gyroscope is mounted, the sequence of angles is fixed. The necessary Euler angle sequence can, however, be determined by inspection.

The pick-off angles of the gyroscope uniquely determine the orientation of the missile. We can demonstrate this very easily in the following manner. The angular velocity vector can be represented as:

$$\vec{\omega} = P\vec{b}_1 + Q\vec{b}_2 + R\vec{b}_3 = \dot{\theta}_g\vec{b}_2 + \dot{\phi}_g\vec{\xi}_{11} + \dot{\psi}_g\vec{\xi}_3$$

where the subscript "g" represents the gyroscopic pick-off angles. The gyroscopic pick-off angles are the integrals of the following quantities:

$$\begin{aligned}\dot{\theta}_g &= Q\\ \dot{\phi}_g &= P\vec{b}_1\cdot\vec{\xi}_{11} + R\vec{b}_3\cdot\vec{\xi}_{11}\\ \dot{\psi}_g &= P\vec{b}_1\cdot\vec{\xi}_3 + Q\vec{b}_2\cdot\vec{\xi}_3 + R\vec{b}_3\cdot\vec{\xi}_3\end{aligned}$$

The integrals of the equations show that the pick-off angles are unique since they are determined entirely by the orientation of the missile in space.

In diagram "I", the sequence (as can be seen by inspection) is a yaw angle (with reference to the spinner) then a roll angle followed by a pitch angle. In diagram "II", the sequence is a roll angle (corresponding to the spinner) followed by a yaw angle and then by a pitch angle. These two sequences are from the "inside out". We can write the two transformation equations in matrix notation as:

$$(\vec{b}_i) = M(\theta)M(\phi)M(\psi)(\vec{\xi}_i) \equiv \mathrm{I}$$

$$(\vec{b}_i) = M(\theta)M(\psi)M(\phi)(\vec{\xi}_i) \equiv \mathrm{II}$$

Where the M's represent transformation matrices in the indicated modes.

Let us determine the transformation for the two types of gyros illustrated. For diagram "I", we will indicate the angles by writing subscript "V" for each angle. In the second one we will use a subscript "H" for each angle. These subscripts mean that the first gyroscope is "vertically fixed" and the second is "horizontally fixed". That is, the spinner position is vertical or horizontal, respectively. From the vantage point of our previous knowledge, we can choose the intermediate matrices for pitch, yaw, and roll. We will indicate these matrices below:

$$M(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$M(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$$

$$M(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Having these three matrices available and choosing the subscript "v" for the first transformation, and the subscript "H" for the second transformation we obtain:

$$\begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix} = \begin{pmatrix} C\psi_H C\theta_H & S\psi_H & -C\psi_H S\theta_H \\ -C\phi_H S\psi_H C\theta_H & C\phi_H C\psi_H & C\phi_H S\psi_H S\theta_H + S\phi_H C\theta_H \\ S\phi_H S\psi_H C\theta_H + C\phi_H S\theta_H & -S\phi_H S\psi_H & -S\phi_H S\psi_H S\theta_H + C\phi_H C\theta_H \end{pmatrix} \begin{pmatrix} \vec{S}_{H1} \\ \vec{S}_{H2} \\ \vec{S}_{H3} \end{pmatrix}$$

$$\begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix} = \begin{pmatrix} S\psi_v S\phi_v S\theta_v + C\psi_v C\theta_v & S\psi_v C\phi_v & S\psi_v S\phi_v C\theta_v - C\psi_v S\theta_v \\ C\psi_v S\phi_v S\theta_v - S\psi_v C\theta_v & C\psi_v C\phi_v & C\psi_v S\phi_v C\theta_v + S\psi_v S\theta_v \\ C\phi_v S\theta_v & -S\phi_v & C\phi_v C\theta_v \end{pmatrix} \begin{pmatrix} \vec{S}_{v1} \\ \vec{S}_{v2} \\ \vec{S}_{v3} \end{pmatrix}$$

Now that we have the actual values of the transformation matrices, we will rewrite the last two transformation matrices in a slightly easier to handle notation. Let the following two matrices represent the "vertical" and "horizontal" matrices respectively:

$$\begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix} = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \begin{pmatrix} \vec{S}_{V1} \\ \vec{S}_{V2} \\ \vec{S}_{V3} \end{pmatrix}$$

$$\begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix} = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix} \begin{pmatrix} \vec{S}_{H1} \\ \vec{S}_{H2} \\ \vec{S}_{H3} \end{pmatrix}$$

The spinner has special significance. The orientation of the spinner axis remains fixed throughout the time that the gyroscope is in operation. (This is not quite true since we are assumming that the earth is flat, and the orientation of the gyro will remain fixed with respect to space rather than the earth's surface. It will appear that the direction of the vectors changes by an angle equal to the change of the normal vector of the earth's surface due to its curvature. However, for this discussion we will assume this angle is small enough to ignore). If the missile is originally "on the ground" and its vectors, the body axes, are aligned parallel to the ground axes, we can make the following statement:

$$\vec{S}_{V3} = \vec{g}_3$$

$$\vec{S}_{H3} = \vec{g}_1$$

That is, the third axis of the vertically fixed gyroscope always points in the direction of the third ground axis, and the third axis of the horizontally fixed gyroscope always points in the same direction as the first ground axis. Let us now write the transformation matrix which we originally had between the ground and body system in short hand notation as follows:

$$\begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} \vec{g}_1 \\ \vec{g}_2 \\ \vec{g}_3 \end{pmatrix}$$

Then we can write, after replacing the $\vec{s}_3$ vector by $\vec{g}_3$ in the first transformation matrix for the gyroscopes:

$$\vec{b}_1 \cdot \vec{g}_3 = V_{13}$$
$$\vec{b}_2 \cdot \vec{g}_3 = V_{23}$$
$$\vec{b}_3 \cdot \vec{g}_3 = V_{33}$$

and from the transformation matrix between body and ground we obtain:

$$\vec{b}_1 \cdot \vec{g}_3 = a_{13}$$
$$\vec{b}_2 \cdot \vec{g}_3 = a_{23}$$
$$\vec{b}_3 \cdot \vec{g}_3 = a_{33}$$

so we can write:

$$V_{13} = a_{13}$$
$$V_{23} = a_{23}$$
$$V_{33} = a_{33}$$

In the transformation matrix for the "II" gyroscope, we can replace $\vec{s}_1$ by $\vec{g}_1$. This gives us the three equations:

$$\vec{b}_1 \cdot \vec{g}_1 = H_{11}$$
$$\vec{b}_2 \cdot \vec{g}_1 = H_{21}$$
$$\vec{b}_3 \cdot \vec{g}_1 = H_{31}$$

and from the transformation matrix between body and ground we obtain:

$$\vec{b}_1 \cdot \vec{g}_1 = a_{11}$$
$$\vec{b}_2 \cdot \vec{g}_1 = a_{21}$$
$$\vec{b}_3 \cdot \vec{g}_1 = a_{31}$$

Therefore, we can write:

$$H_{11} = a_{11}$$

$$H_{21} = a_{21}$$

$$H_{31} = a_{31}$$

and we now write for the transformation matrix between body and ground:

$$\begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix} = \begin{pmatrix} H_{11} & a_{12} & V_{13} \\ H_{21} & a_{22} & V_{23} \\ H_{31} & a_{32} & V_{33} \end{pmatrix} \begin{pmatrix} \vec{g}_1 \\ \vec{g}_2 \\ \vec{g}_3 \end{pmatrix}$$

Utilizing the properties of orthogonal matrices explained at the beginning of this section, we can solve for the middle column of the above matrix. This is:

$$a_{12} = - \begin{vmatrix} H_{21} & V_{23} \\ H_{31} & V_{33} \end{vmatrix} = H_{31} V_{23} - H_{21} V_{33}$$

$$a_{22} = \begin{vmatrix} H_{11} & V_{13} \\ H_{31} & V_{33} \end{vmatrix} = H_{11} V_{33} - H_{31} V_{13}$$

$$a_{32} = - \begin{vmatrix} H_{11} & V_{13} \\ H_{21} & V_{23} \end{vmatrix} = H_{21} V_{13} - H_{11} V_{23}$$

We have now completely determined the angles between the body and the ground frame in terms of the pick-off angles of the gyros.

In the transformations between the gyro system and the body system, the vertically fixed gyro used the sequence: yaw, roll, and pitch. The horizontally fixed gyro used the sequence roll, yaw, and pitch. It was pointed out, when we were performing the transformation between body and ground, that the second angle must never approach very close to 90°. This was because, when determining the rollrates of the missile, it was necessary to use the secant of the second angle. If the second angle ever approached 90°, then the value of the

entire term would become indeterminant. An analogous situation exists with the gyro angles. If the second angle goes to 90° then a situation arises called "gimbal lock". Therefore, in the vertical-fixed gyro, the roll must never approach 90°, and in the horizontal - fixed gyro, the yaw must never approach 90°. If they do, and gimbal lock occurs, then it is no longer possible to know the orientation of the missile. The physical reaction of the gyros is often a complete 180° turn of the entire gyro setup -- and a new set of orientation vectors.

With our previous knowledge of transformation between systems, we were able to obtain the complex transformation equations for utilization of the gyro pick-off angles very readily. If we tried to visualize the complicated interrelationships of the pick-off angles to each other and also the interrelationships to the transformation angles between the ground and the body, it would have been impossible to solve the equations. But with the utilization of coordinate systems and transformation matrices between these systems, we can find the values in a very simple manner. It is not until we try to evaluate the terms derived, such as the V's, the H's, and the a's of the matrices that the equations begin to look complicated. Even then they are not difficult, they merely have a great many terms.

A separate section will not be introduced for an explanation of stable platforms. The main difference between the representation of stable platforms and gyroscopes is that a stable platform (which is stabilized by means of gyroscopes), has available three pick-off angles. In our analysis, using gyroscopes, we required two of them in order to determine the three angles. Had we decided to pick-off the angle for the spinner and evaluate its size (by integration of the spinner angle), we could have done a very similar thing as a stable platform would do. As a matter of fact we "pretended" in our transformation between the gyro system and the body system that we actually had this fixed direction angle. We did not use this fixed angle, but determined the orientation angles of the body with respect to the ground from only the outer two pick-off angles. If anyone who is using these notes finds it necessary to determine the equations relating a missile system to a stable platform, then he need only remember that he transforms from the stable platform to the body system through three angles. And also there is no second gyroscope to worry about. Therefore, the stable platform is more simple, in essence, than the gyroscopes which we just investigated.

GUIDANCE

Included in the accompanying sections are typical examples of guidance and some of the nomenclature one is likely to encounter.

GUIDANCE AND CONTROL SYSTEM FOR A TYPICAL RADIO COMMAND MISSILE SYSTEM

By: Harry McJunkin

In such a system, the chief inputs into the ground computer are the position coordinates of the target and of the missile. These coordinates, for the typical case, are range, elevation angle and azimuth angle as measured by radar instrumentation.

The ground computer transforms these coordinates into rectangular coordinates in a common frame. The target coordinates are time differentiated to determine present target velocity, then the time of missile flight to a predicted intercept of the target is determined from use of present target position, present target velocity, and stored missile capability curves. Of course the computations are modified to take care of the offset transient time needed for most effective use of the warhead.

Other functions necessary for efficient steering of the missile are generated from their established dependence on the coordinates of the predicted burst point. These functions include such items as initial dive order, glide bias, and missile velocity correction. The dive order is composed of acceleration orders to the missile causing it to turn from its initial nearly vertical trajectory (given during the boost phase which is not controlled) into a semi-parabolic path towards the predicted intercept point. The glide bias orders are commands, separate from steering orders, causing the missile to always have a certain fraction of 1 g acceleration in a nearly vertical direction. The missile velocity correction added to "present" missile velocity during steering provides more accurate determination of time-to-go and consequently of the steering orders.

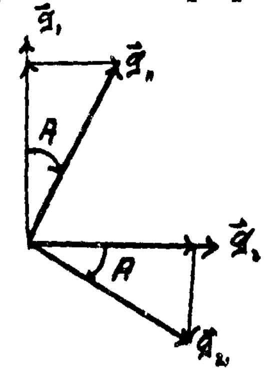
The steering orders are determined, essentially, as error quantities proportional to the differences between the <u>required</u> closing velocity coordinates, and the <u>actual</u> closing velocity coordinates. The <u>required</u> closing velocity coordinates are the differences between missile position coordinates and corresponding target position coordinates, divided by the time-of-flight to the predicted intercept. The "present" target altitude is modified to furnish an aiming point above the target for most of the trajectory, and thus provide greater missile end-game maneuverability.

During a large part of the trajectory, the remaining time-of-flight is continually recalculated in terms of present missile and target velocities. This function, called time-to-go, replaces the function initially termed time-of-flight. The velocities of missile and target, as well as target accelerations, are determined by analog type differentiation devices. The <u>actual</u> closing velocity coordinates are found by simply subtracting the <u>target</u> velocity coordinate from corresponding missile velocity coordinates. Of course these closing velocity coordinates are modified by the missile velocity correction function, and by present target acceleration components.

The error quantities, described above, are basically velocity differences in the fixed rectangular ground frame associated with either the launching point, the missile tracking radar, or the target tracking radar. These quantities are converted into components in a rectangular frame nearly parallel to the "pseudo" missile frame, -- a fixed frame attached to the missile.

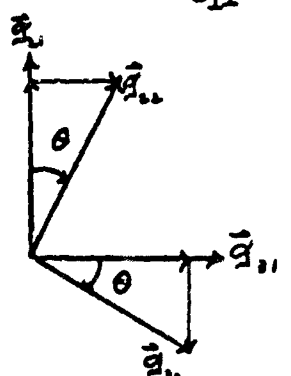
This conversion is accomplished by a yaw, pitch, yaw sequence as follows:

First rotate about $\vec{g}_3$ through angle A, where A is the azimuth angle of the computed intercept point.



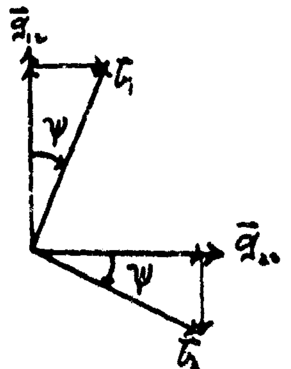
$$\vec{g}_{11} = \vec{g}_1 \cos A + \vec{g}_2 \sin A$$
$$\vec{g}_{21} = \vec{g}_1 \sin A + \vec{g}_2 \cos A$$
$$\vec{g}_{31} = \vec{g}_3$$

Next rotate about $\vec{g}_{11}$ through a "climb angle" θ, (θ will be derived later):



$$\vec{g}_{12} = \vec{g}_{11}$$
$$\vec{g}_{22} = \vec{g}_{21} \cos\theta + \vec{g}_{31} \sin\theta$$
$$\vec{g}_{32} = -\vec{g}_{21} \sin\theta + \vec{g}_{31} \cos\theta$$

Finally rotate about $\vec{g}_{32}$ through a "turn angle" ψ, (ψ will be derived later):

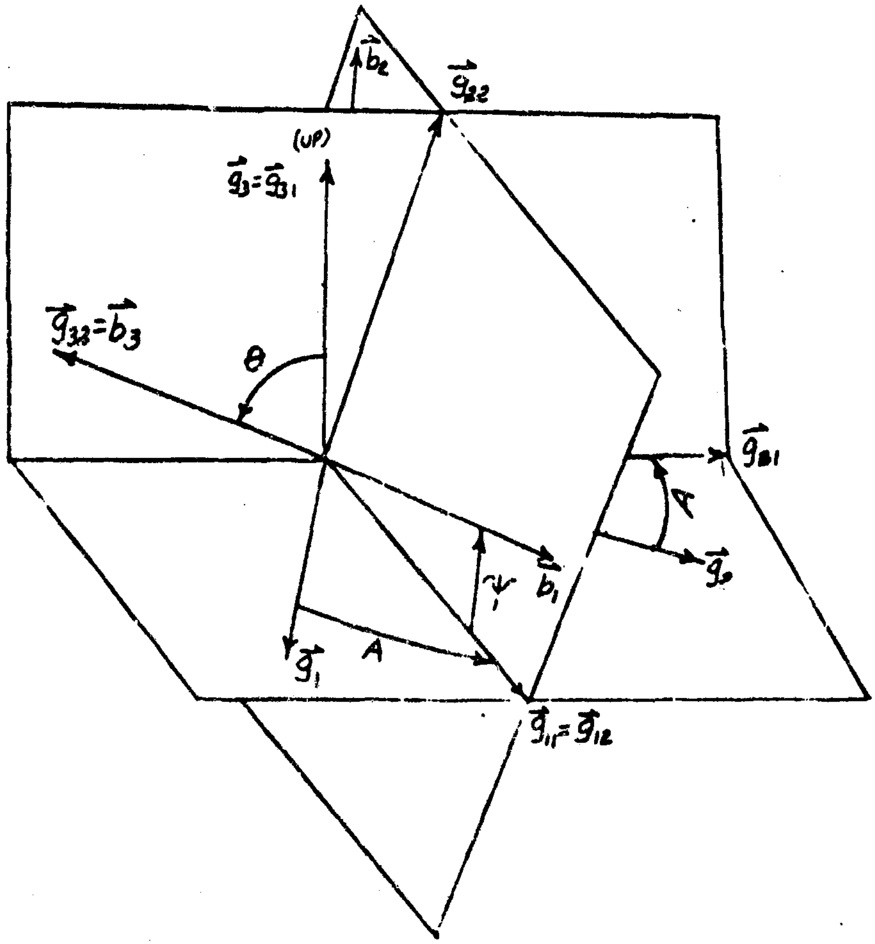


$$\vec{b}_1 = \vec{g}_{12} \cos\psi + \vec{g}_{22} \sin\psi$$
$$\vec{b}_2 = -\vec{g}_{12} \sin\psi + \vec{g}_{22} \cos\psi$$
$$\vec{b}_3 = \vec{g}_{32}$$

Combining these in matrix form we get the transformation matrix below for the pseudo-missile frame:

$$\begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{b}_3 \end{pmatrix} = \begin{pmatrix} \cos\psi\cos A - \sin\psi\cos\theta\sin A & \cos\psi\sin A + \sin\psi\cos\theta\cos A & \sin\psi\sin\theta \\ -\sin\psi\cos A - \cos\psi\cos\theta\sin A & -\sin\psi\sin A + \cos\psi\cos\theta\cos A & \cos\psi\sin\theta \\ \sin\theta\sin A & -\sin\theta\cos A & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{g}_1 \\ \vec{g}_2 \\ \vec{g}_3 \end{pmatrix}$$

These rotations are pictured in the following diagram:



At this point the two angles, θ and ψ, are not known. Angle A is known since the coordinates of the intercept point have presumably already been computed.

If we assume there is no angle of attack we may say

$$\vec{V}_M = V\vec{b}_2 = V_x\vec{g}_1 + V_y\vec{g}_2 + V_z\vec{g}_3$$

where $V = |\vec{Vm}| = \sqrt{V_x^2 + V_y^2 + V_z^2}$

taking the dot products with g_1, g_2 and g_3, respectively, we obtain:

$$V\vec{b_2}\cdot\vec{g_1} = V_x = V(-\sin\psi\cos A - \cos\psi\cos\theta\sin A)$$

$$V\vec{b_2}\cdot\vec{g_2} = V_y = V(-\sin\psi\sin A + \cos\psi\cos\theta\cos A)$$

$$V\vec{b_3}\cdot\vec{g_3} = V_z = V\cos\psi\sin\theta$$

If we multiply (1) by cos A and (2) by sin A and add we obtain:

$$V_x\cos A + V_y\sin A = -V\sin\psi$$

$$\text{or } \sin\psi = -\left(\frac{V_x\cos A + V_y\sin A}{V}\right)$$

and if we multiply (1) by sin A and (2) by cos A and subtract (1) from (2) then:

$$(4) \qquad -V_x\sin A + V_y \mathrm{CO} A = V\cos\psi\cos\theta$$

Multiplying (4) by sin θ and (3) by cos θ and subtracting (4) from (3) we get:

$$V_x\sin A\sin\theta - V_y\cos A\sin\theta + V_z\cos\theta = 0$$

and dividing by cos θ:

$$\tan\theta\,(V_x\sin A - V_y\cos A) + V = 0$$

$$\text{or } \tan\theta = \frac{-V_z}{V_x\sin A - V_y\cos A}$$

Since the missile is stabilized at a known angle of roll about its longitudinal axis; and with respect to the plane containing the velocity vector, the conversion of error quantities into the "pseudo" missile frame can be completed. The lateral control directions of the missile nearly coincide with the lateral axes of the pseudo missile frame.

Time-to-go is continually determined during a large part of flight as previously mentioned. This is done in such a way that only lateral commands to the missile are different from zero, since normally there is negligible control in the longitudinal direction through use of elevon control.

Finally the lateral error quantities, which have dimensions of velocity, are converted to dimensions of acceleration through division by time-to-go; then these are scaled to the proper order magnitudes through multiplication by a constant factor. The orders are next sent to command circuits which convert the orders to coded electrical pulse groups. These groups may be generated intermittently and alternatively at audio frequency. The radar transmitter accepts these pulsed orders and transmits the information via an electromagnetic link to the missile receiver.

The missile receiver recognizes properly coded signals, reconverts these to pulse groups, and sends these to a signal data converter. Each time a pulsed group is recognized, a hold-off signal may be issued to burst control circuits and an activating signal sent to the missile beacon. If insufficient autopilot control signals are received the hold-off pulse may be discontinued and the detonation circuits become operative.

The signal data converter usually recovers in dc analog form the ground computer acceleration orders to the missile. These orders are sent to the autopilot control systems. The lateral acceleration control systems are usually identical in form, each exercising control in a plane perpendicular to the hinge line of the control fins of the system. For a supersonic missile the lateral control direction is nearly normal to the longitudinal axis of the missile.

A lateral control system may be comprised of a steering amplifier with associated input mixing networks, a hydraulic system to control servo power an actuator assembly driven by this power, a mechanical linkage to the control elevons, and several feedback signals to the steering amplifier. These feedbacks may include signals representing the missile acceleration achieved in the desired direction, missile angular velocity in the control plane, and average control fin position. The command signals are compared to these feedback signals which are developed by missile flight instruments. The networks associated with the steering amplifier, and into which the orders and feedbacks are inputs, compare somewhat continuously the present response conditions of the missile to the present steering command. The result is a smooth response to orders, and a preservation of missile stability throughout the flight.

The roll control system operates very much like one of the lateral control systems, excepting that the roll control signal arises from the missile rolling more or less than the desired roll stabilization angle, as described above. Useful feedback signals may be angular roll velocity and roll elevon position.

Efficient missile systems may utilize lateral control elevons to achieve roll control as well as lateral acceleration control. For such systems the mechanical linkages are arranged to that average control elevon position is correct for the desired lateral acceleration, while the differential between opposite fin deflections is correct for achieving roll control.

The following diagrams illustrate a typical radar-command roll-stabilized system as described above.

The following diagrams illustrate a typical radar command roll stabilized system as above described:

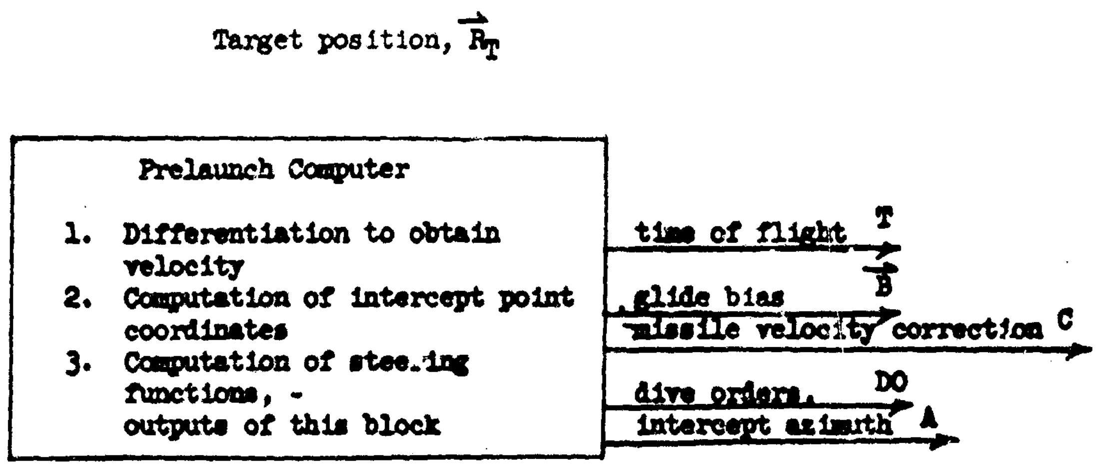


FIGURE 1
Prelaunch Ground Computations

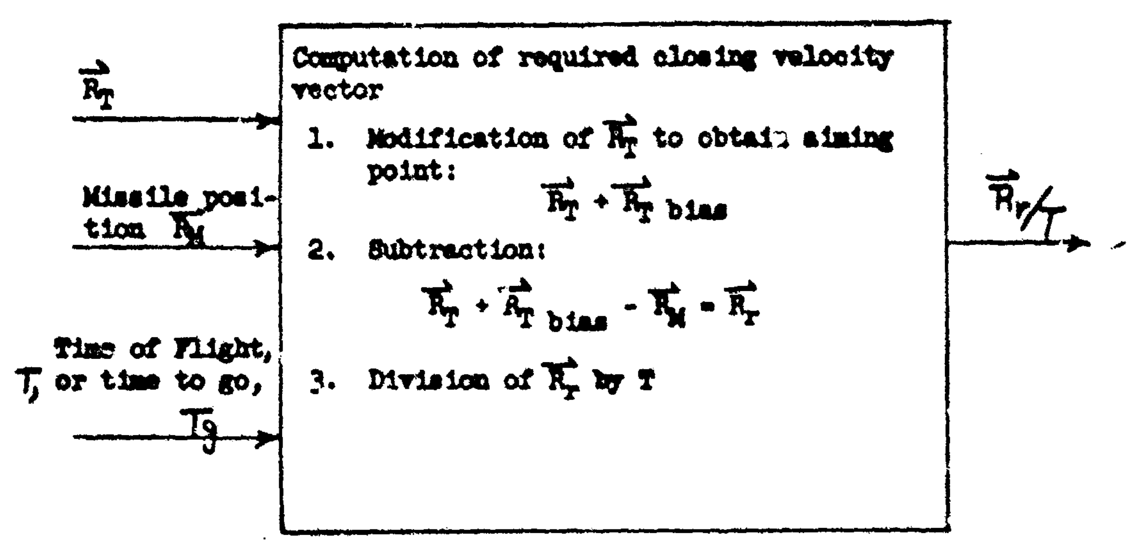


FIGURE 2
Computation of Required Closing Velocity Vector

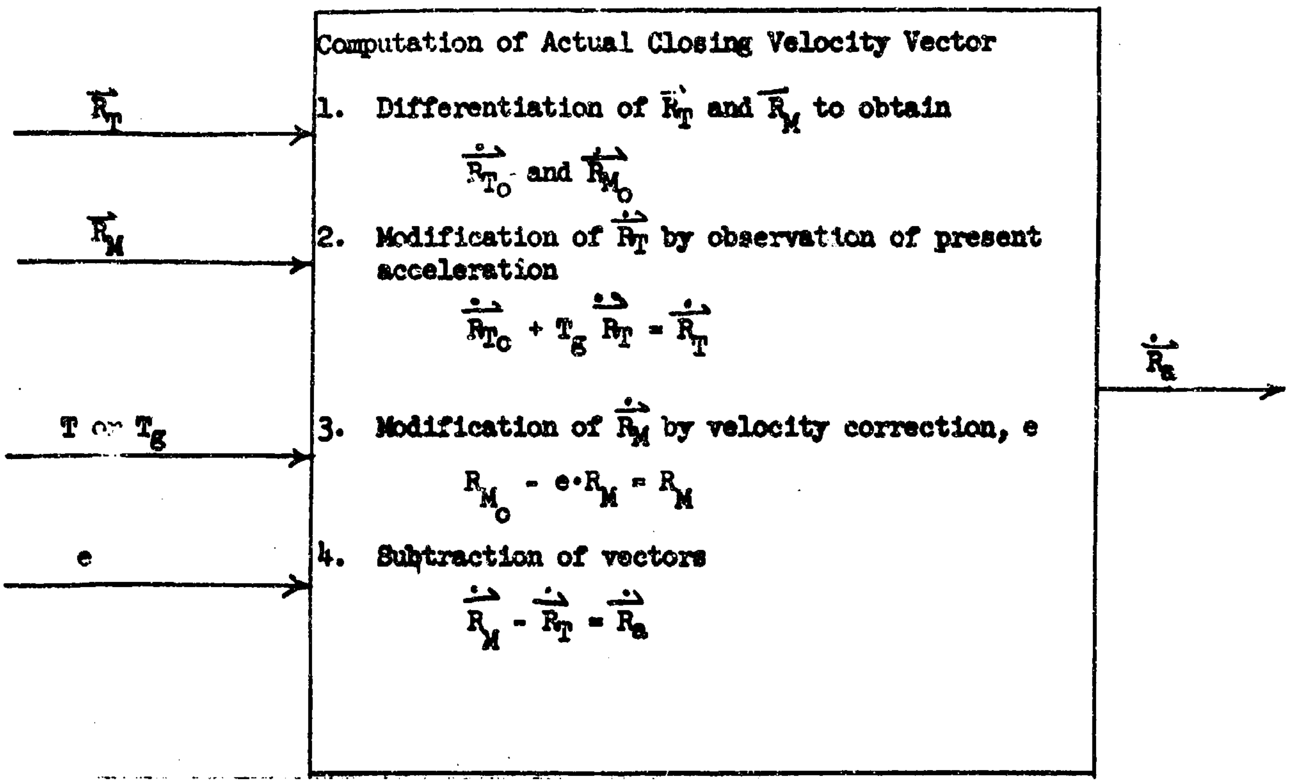


FIGURE 3

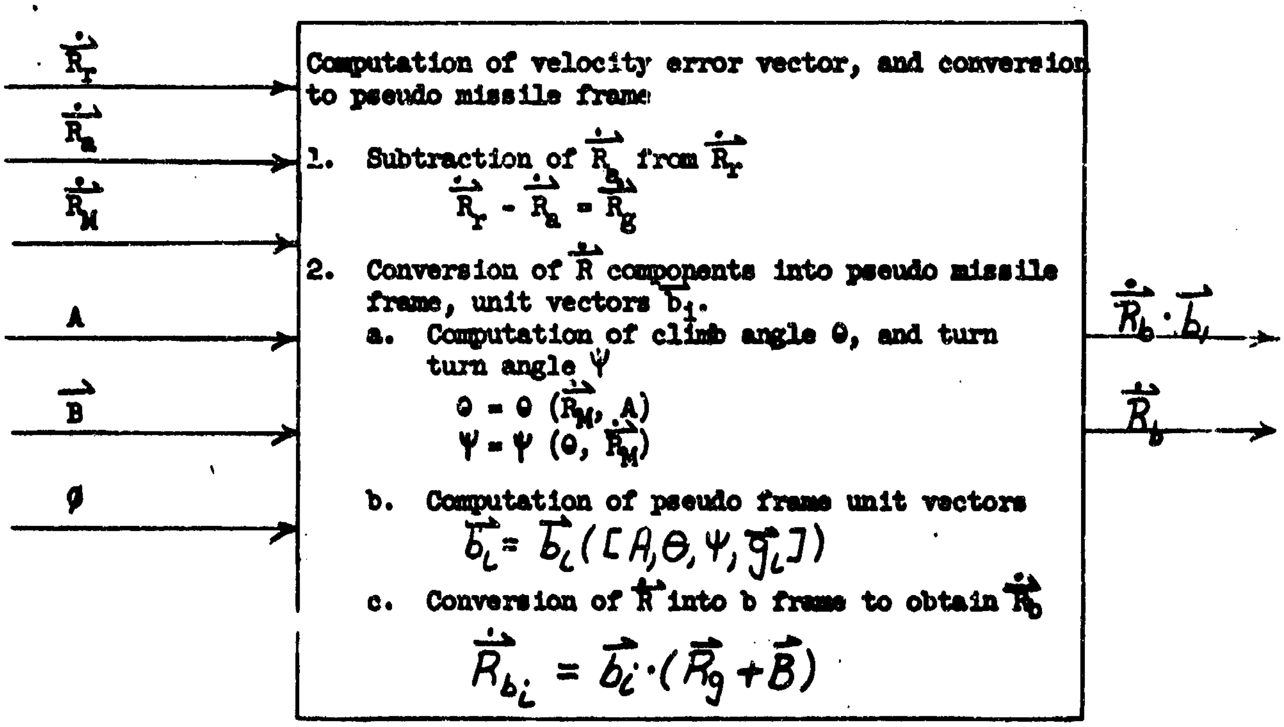


FIGURE 4

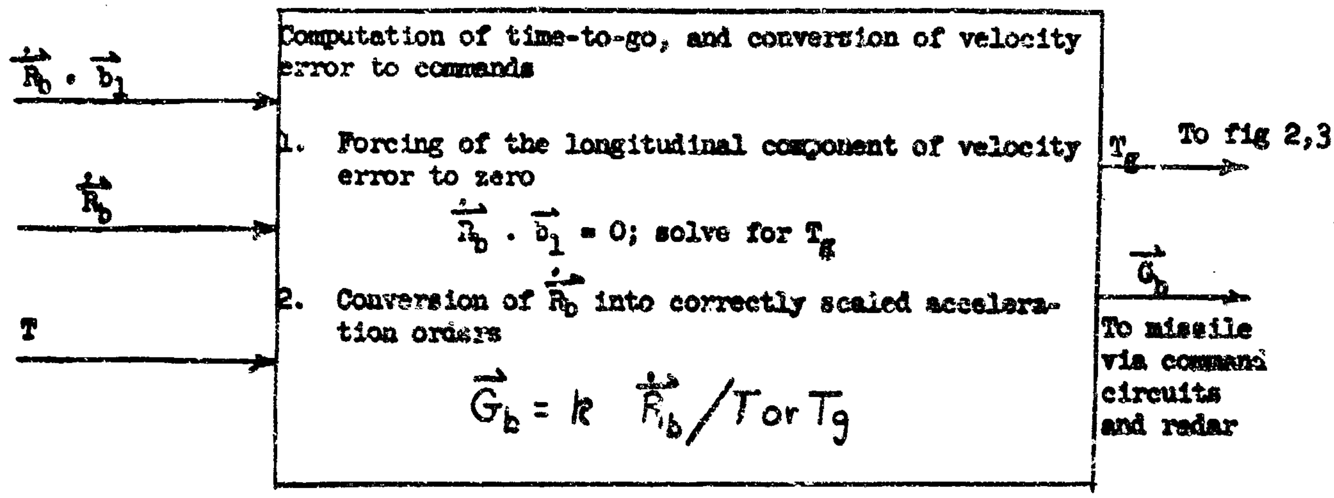


FIGURE 5

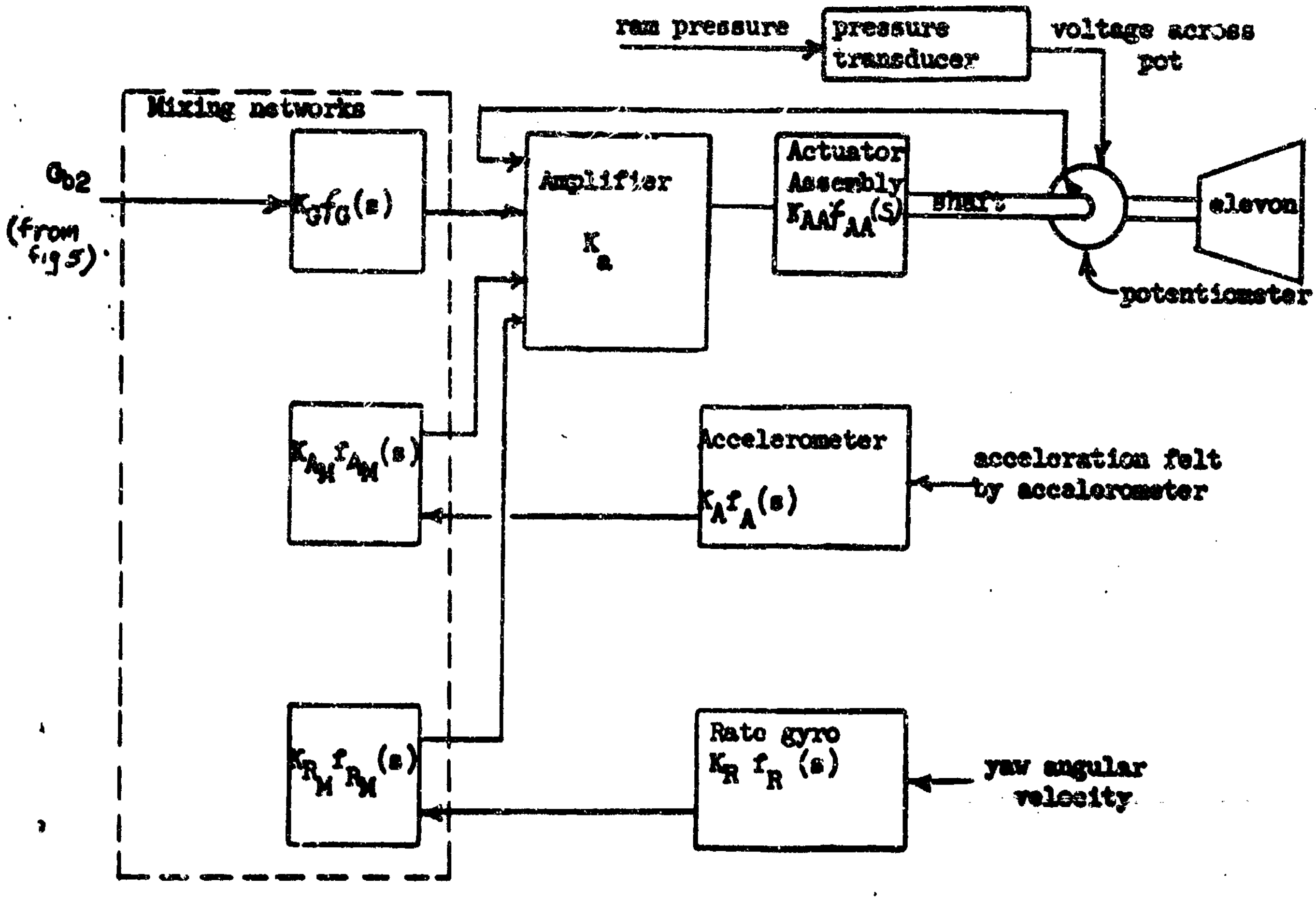


FIGURE 6

SEMI-ACTIVE HOMING GUIDANCE SYSTEM

By Merle Lesoing

In a semi-active homing guidance system, a proposed target is radiated with RF (radio frequency) energy from some source external to the missile, and the missile utilizes the energy reflected off of the target to derive guidance information. For example, a ground based CW (continuous wave) illuminator tracks the proposed target and radiates it with RF energy. The missile receives the reflected energy through a gimballed, forward-looking antenna dish. The antenna has a nutating beam which allows the missile to derive tracking errors which cause the dish to always point toward the target.

In order to detect and to continue tracking one specific target out of many possible targets that may be within the beam width of the Illuminator, the internal circuitry may be equipped with a very narrow velocity "gate" which it uses to resolve a single target. Thus the antenna tracking errors can then be derived for this one target. The velocity gate makes use of the doppler shift, which is the shift in frequency of a signal due to the radial velocity between the transmitter and the reflector. A "rearward-looking" antenna in the missile heterodynes a sample of the transmitted energy from the Illuminator with the reflected target signal received at the front antenna. The difference in frequency, called the "doppler", is proportional to the radial or "closing" velocity of the target with respect to the missile.

The range of "doppler frequencies" of all the possible targets that might occur for this type of guidance can be determined. The band is quite narrow. The velocity gate is swept through the possible frequencies until a "doppler" appears in the gate. The antenna then "locks on" to the target which has this doppler frequency and obtains pointing errors from this one target only.

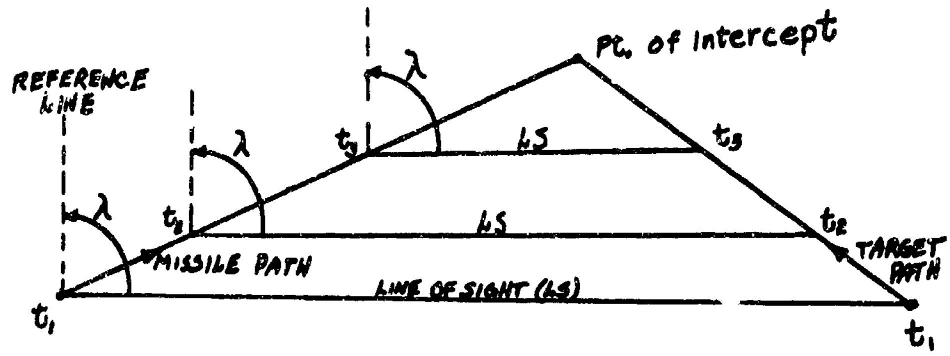
When the missile is locked on to a particular target, then the direction of the target is known. Using the known direction of the antenna, control information for the missile is derived so as to produce a trajectory which will intercept the target.

The homing-type guidance can employ several alternative methods for interception. The most simple guidance, of course, would maneuver the missile so that it is always headed toward the target. The drawbacks to this procedure are immediately obvious. If the target maneuvers, the missile might expend so much of its energy correcting its course, that it is unable to maintain sufficient speed to reach intercept. Also, if the target has a high speed, the missile would have to chase it, again expending a great deal of energy.

Generally, a type of guidance is employed which, utilizing the target velocity and direction, derives a predicted intercept point; the missile is directed toward this point. This method is called proportional navigation. By proportional navigation we mean that the rate of change of missile heading

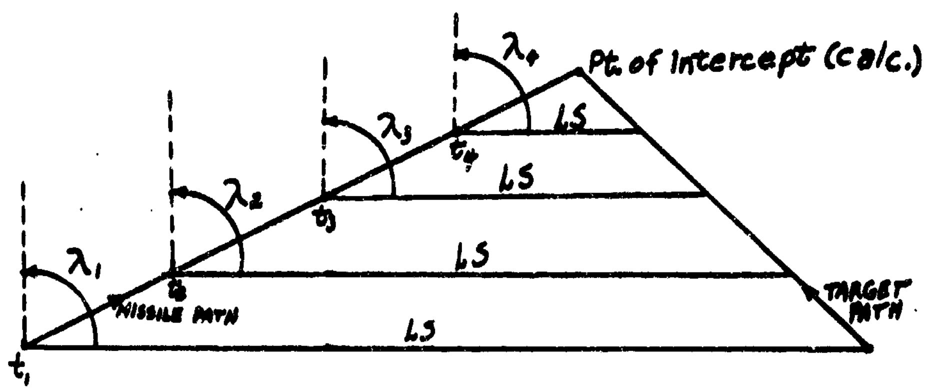
(that is, how fast the missile must turn) must be directly proportional to the rate of rotation of the line-of-sight from the missile to the target.

To appreciate the application of this concept, consider the figure below. In this figure, the missile and target position at equal time intervals, t_i, are depicted in two dimensions.



It can be seen that the line-of-sight angle, λ, does not change--hence the target and missile are on a collison course.

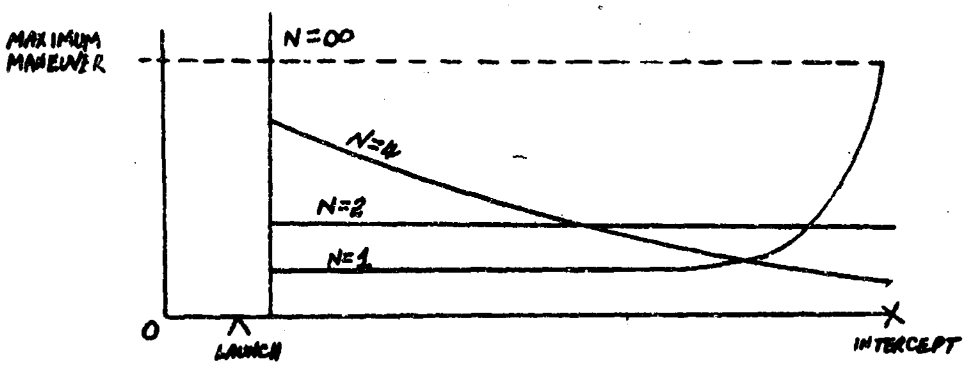
The next figure shows the missile reaction if a line-of-sight rate exists. The missile then attempts to correct its own motion so that it will follow a constant heading-angle that will result in intercept.



Here we can see that the line-of-sight angle, λ, is changing. The missile must turn at a rate, proportional to the line-of-sight rate, in such a way that the change of λ is decreased to zero. When this occurs, the missile is oriented along a collision course. Expressed mathematically this relation becomes:

$$\omega_H = N\omega_{LS}$$

where ω_H is the rate of turn of the missile heading (that is, the missile velocity vector) ω_{Ls} is the rate of turn of the line-of-sight of the missile to the target, and N is the proportionality constant. The choice of N depends upon how much response is needed for the maneuver and whether the missile can survive the maneuver without breaking. The next figure indicates acceleration as a function of the distance from the missile to the point of intercept for various values of N:



For a medium speed target, N = 4 appears to be a highly suitable choice for the navigation ratio.

If medium speed targets were the only kind we expected, the choice of N = 4 for the navigation ratio would be adequate. However, we need greater maneuverability for high speed targets, but we do not wish to overcorrect for slower targets. To compensate for various target speeds an "effective" navigational ratio, N', is defined such that

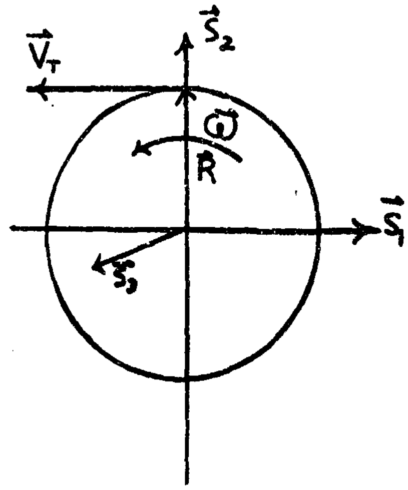
$$N' = \frac{V_C}{V_M} N,$$

where V_C is the closing velocity of the target to the missile and V_M is the missile velocity. Thus, the closing velocity governs the size of the ratio, V_C/V_M, providing for optimum maneuverability for the missile.

The proportional navigation ratio is used to determine acceleration commands to the missile. This can be seen from the following analysis. We know that a particle following a curved path will have a tangential velocity, $\vec{V}_T$, equal to the cross product of the angular velocity and the radius-of-curvature vector. Thus:

$$\vec{V}_T = \vec{\omega} \times \vec{R}$$

For example, if the particle is following a circular orbit, then:



$$\vec{V}_T = \vec{\omega} \times \vec{R} = \begin{vmatrix} \vec{S}_1 & \vec{S}_2 & \vec{S}_3 \\ 0 & 0 & \omega \\ 0 & R & 0 \end{vmatrix} = -\omega R \vec{S}_1$$

The centripetal, or lateral, acceleration is the change of $\vec{V}_T$ with respect to the angular rate. Thus, we have:

$$\vec{A}_L = \vec{\omega} \times \vec{V}_T = \vec{\omega} \times (\vec{\omega} \times \vec{R}) = \begin{vmatrix} \vec{S}_1 & \vec{S}_2 & \vec{S}_3 \\ 0 & 0 & \omega \\ -\omega R & 0 & 0 \end{vmatrix} = -\omega^2 R \vec{S}_2$$

or

$$\vec{A}_L = V_T \, \omega \, \vec{S}_2$$

Applying this formula to the total velocity, $\vec{V}_M$, of the missile and the rate of turn of the missile, $\vec{\omega}_M$, we have:

$$\vec{A}_L = \vec{\omega}_M \times \vec{V}_M .$$

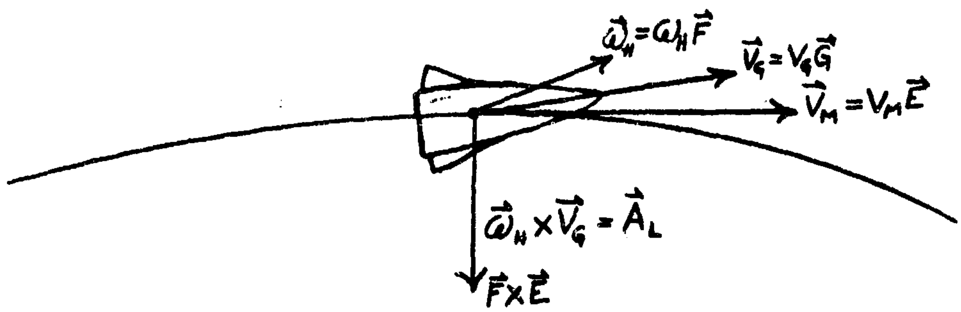
If we assume that

$$\vec{V}_M = V_M \vec{E} \qquad \text{and} \qquad \vec{\omega}_M = \omega_M \vec{F}$$

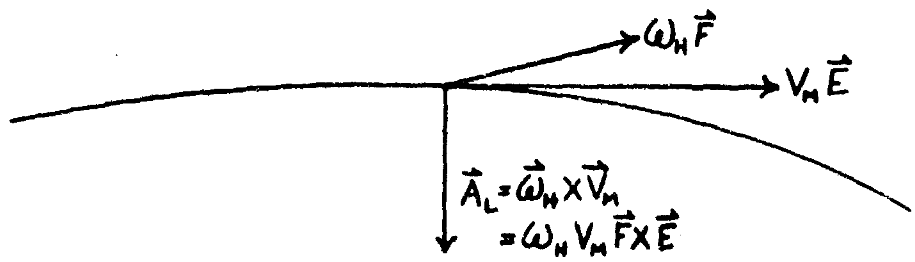
where $\vec{E}$ and $\vec{F}$ are unit vectors for the velocity and angular rate respectively, we can write:

$$\vec{A}_L = \omega_M V_M \vec{F} \times \vec{E}$$

For the particular problem under consideration, we are interested in determining an acceleration which will change the direction of the missile so that the line-of-sight rate will go to zero. The acceleration desired will be directed at right angles to the plane of the rotational vector and the missile heading vector. The rotational velocity (for the desired change) will be the rate of turn of the missile heading ω_H. The following diagram illustrates this:



For this maneuver (alteration of the heading angle) we can assume that the angles of attack are sufficiently small so that we can replace the heading velocity, $\vec{V}_G$, by the total missile velocity, $\vec{V}_M$. Our desired maneuver, then can be seen from the following illustration:



Calling $\vec{A}_L$ the "command" acceleration, $\vec{A}_{MC}$ we can write:

$$\vec{A}_{MC} = A_{MC} \vec{F} \times \vec{E} = \omega_H V_M \vec{F} \times \vec{E}$$

Substituting this equation into the expression for the proportional navigation ratio gives:

$$\frac{A_{MC}}{V_M} = N'\omega_{SL}$$

Which becomes, using the effective navigational ratio:

$$\frac{A_{MC}}{V_M} = \frac{V_C}{V_M} N\omega_{SL},$$

or

$$A_{MC} = V_C N \omega_{SL}$$

Thus from a measurement of the sight-line turning rate we can derive lateral acceleration commands to the missile.

The only acceleration commands to the missile are:

$$A_{MC} = N V_C \omega_{SL}$$

Since, in the previous derivation, we were only interested in the lateral acceleration, and since our commands are only to be given for lateral acceleration, the approximations we assumed are fully justified.